METR 5113, Advanced Atmospheric Dynamics I
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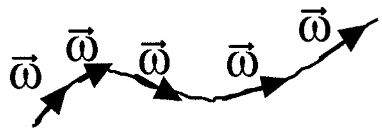
Monday 29 Rocktober 2017 (lecture 29)

2 handouts: Info about exam2, answers to prob set 4.

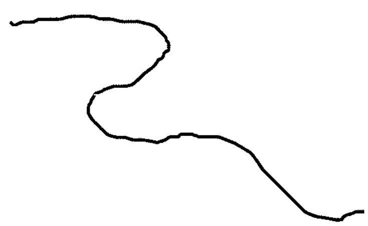
<u>Helmholtz theorem</u> [2nd theorem named after him; see lec 13)]

Helmholtz Th^m is valid for same conditions as Kelvin's Th^m. It says: <u>"Vortex lines move with the flow," that is, vortex lines are material lines.</u>

What does it mean? Consider a vortex line at $t = t_1$:

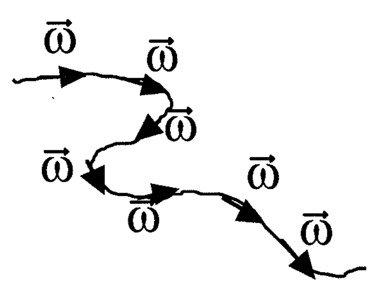


Track the <u>material line</u> coinciding with the vortex line at $t = t_1$. At $t = t_2$, this material line might look like,



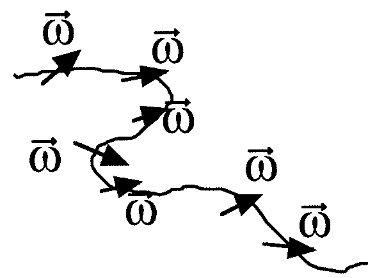
$t = t_2$

Helmholtz Th^m says that since this material line was originally a vortex line, it is still a vortex line. So:



$t = t_2$

If Helmholtz Th^m were not true (e.g., if diffusion or heating is important) then the material line need not be a vortex line after the initial time:



$t = t_2$

Modify Kelvin's Th^m for baroclinic flow -- go back to prev eq^n (*) but this time the pressure integral does not vanish:

$$\frac{D\Gamma}{Dt} = -\oint \frac{1}{\rho} \nabla p \cdot d\vec{l}$$

baroclinic or "solenoidal" term.

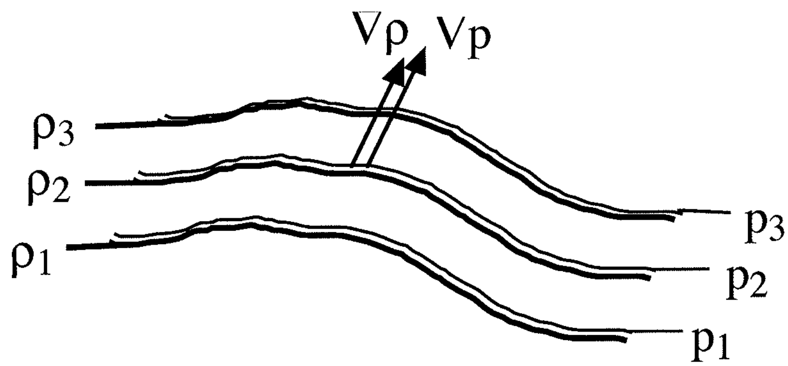
$$= -\int \left[\nabla \times \left(\frac{1}{\rho} \nabla p \right) \right] \cdot \hat{n}\, dA \qquad \text{(using Stokes th}^m\text{)}$$

$$= -\int \left[\nabla \frac{1}{\rho} \times \nabla p + \frac{1}{\rho} \boxed{\nabla \times \nabla p} \right] \cdot \hat{n}\, dA$$

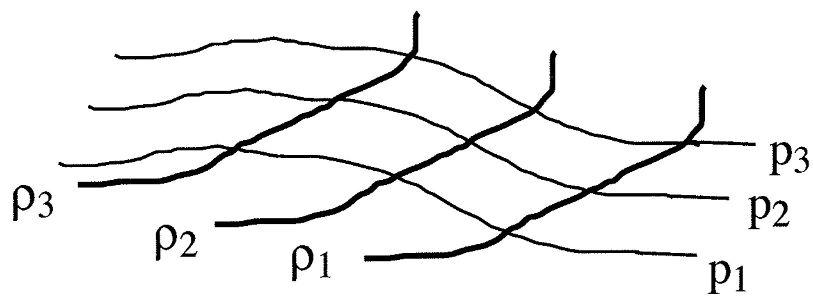
$\downarrow$ use vector identity: $\nabla \times \nabla \text{scalar} = 0$

$$\therefore \boxed{\frac{D\Gamma}{Dt} = \int \frac{1}{\rho^2} (\nabla \rho \times \nabla p) \cdot \hat{n}\, dA}$$

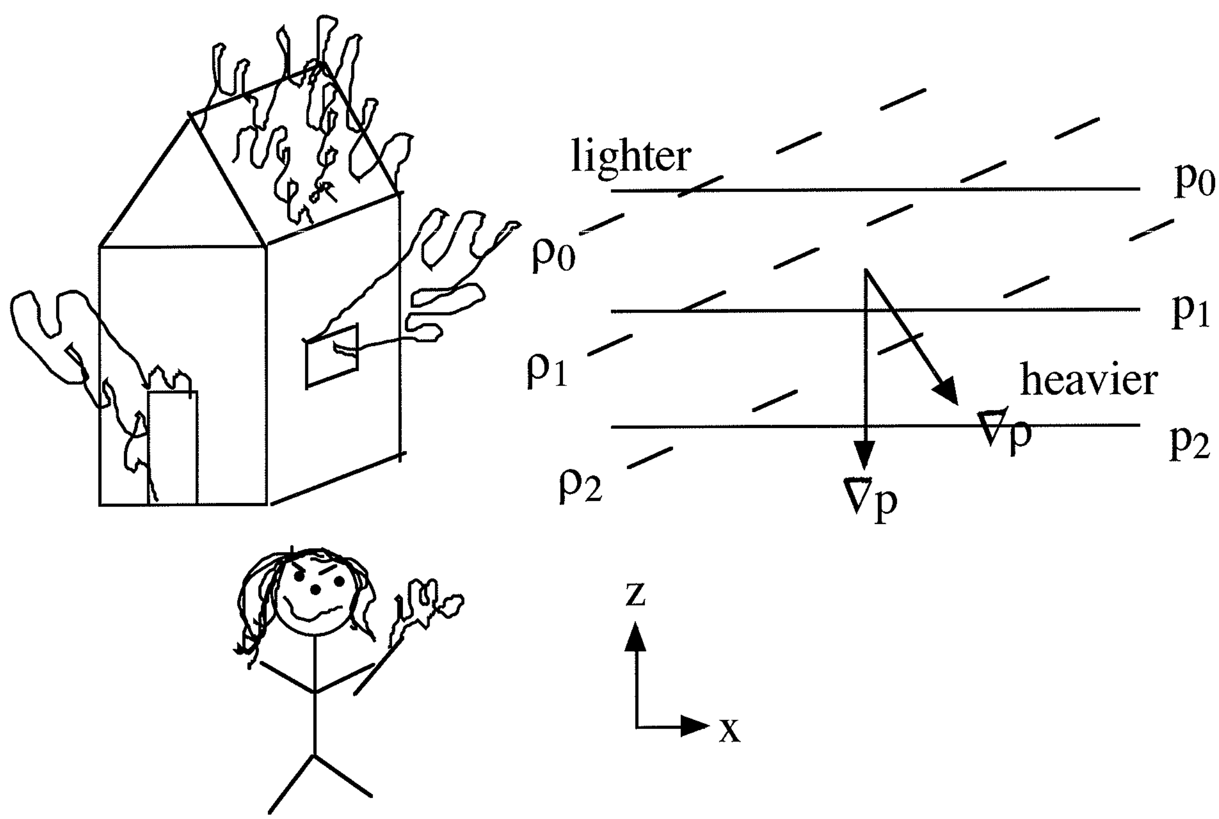
If flow was barotropic then rhs would be 0. Why? Because in barotropic flows, isolines of p and ρ coincide so $\nabla\rho$ is parallel to ∇p so their cross product is 0:



In baroclinic flow, isolines of p, ρ are skewed w.r.t. each other:

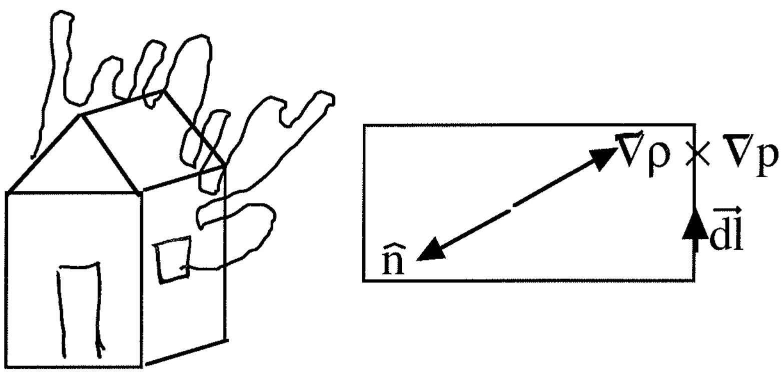


<u>Example of circulation induced in a baroclinic flow:</u>



Pressure is nearly hydrostatic $\therefore$ ∇p points down ($-\hat{k}$ dir^n). Density decreases with height. But due to fire it also decreases toward house. So $\nabla\rho \times \nabla p$ points into page.

Consider a material curve in xz plane next to house. Let $\hat{n}$ point out of page (so $\vec{dl}$ goes around counterclockwise -- to keep interior on left) [could have chosen $\hat{n}$ to point into page and your final answer would have same physical meaning...]

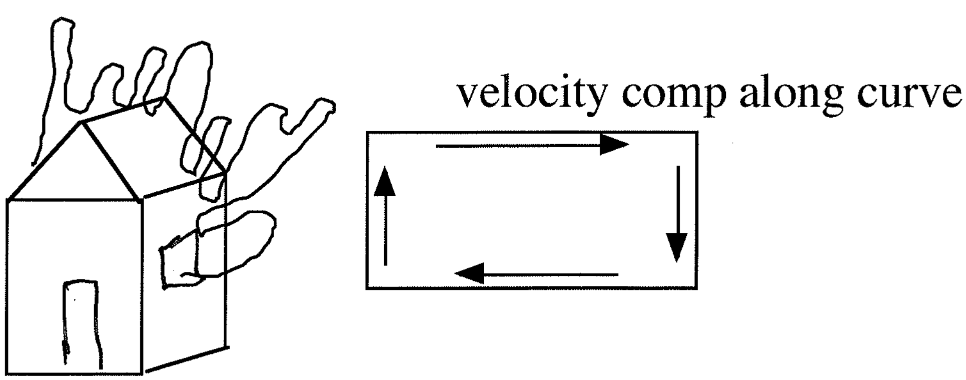


$$\therefore \quad (\nabla\rho \times \nabla p)\cdot\hat{n} < 0$$

$$\therefore \quad \frac{D\Gamma}{Dt} < 0$$

$$\therefore \quad \frac{D}{Dt}\oint \vec{u}\cdot\vec{dl} < 0$$

Net effect is to generate a flow directed in opposite sense to $\vec{dl}$.



Vorticity Equation

- include friction and baroclinicity, but assume flow is incompressible.

Start w/ Navier-Stokes eq^{ns}:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\,\vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} + \boxed{\vec{g}} \dashrightarrow -\nabla \Pi$$

use vector identity: $(\vec{u} \cdot \nabla)\,\vec{u} = \nabla \frac{q^2}{2} + \vec{\omega} \times \vec{u}$

$$\therefore \quad \frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{q^2}{2} + \Pi\right) + \vec{\omega} \times \vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u}$$

Take $\nabla \times$ eq^n. Recall that $\nabla \times \nabla(\text{scalar}) = 0$.

$$\therefore \quad \frac{\partial \vec{\omega}}{\partial t} + \boxed{\nabla \times (\vec{\omega} \times \vec{u})} = -\nabla\frac{1}{\rho} \times \nabla p - \frac{1}{\rho}\underbrace{\boxed{\nabla \times \nabla p}}_{0} + \nu \nabla^2 \vec{\omega}$$

↓use vector identity

$$\nabla \times (\vec{\omega} \times \vec{u}) = \vec{\omega}\underbrace{\boxed{\nabla \cdot \vec{u}}}_{\substack{0 \text{ for} \\ \text{incomp flows}}} - \vec{u}\underbrace{\boxed{\nabla \cdot \vec{\omega}}}_{\substack{0 \text{ for} \\ \text{all flows}}} + (\vec{u} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\vec{u}$$

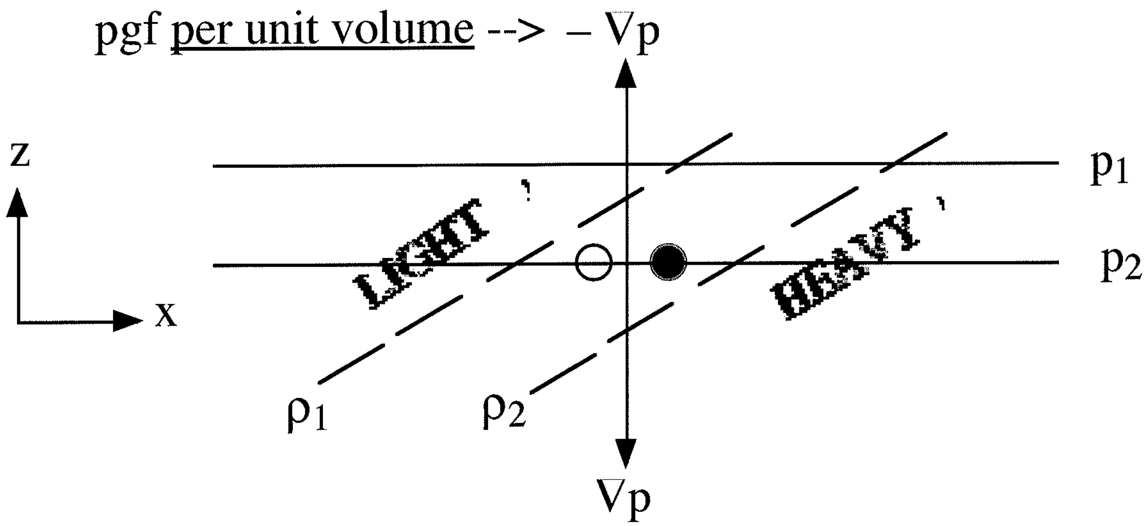
The result is the vorticity eq^n:

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\,\vec{\omega} = (\vec{\omega} \cdot \nabla)\,\vec{u} + \frac{1}{\rho^2}\nabla \rho \times \nabla p + \nu \nabla^2 \vec{\omega}}$$

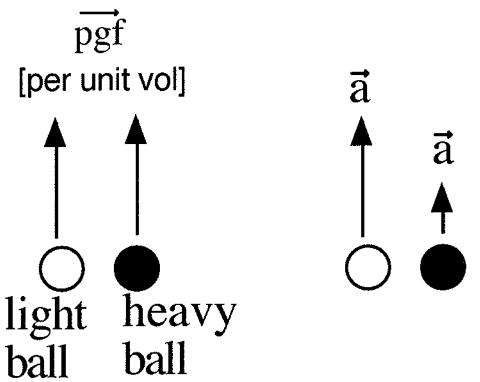
local deriv of vort	advection of vort	tilting and stretching term	baroclinic term	diffusion of vort

Baroclinic term acts mostly in horiz dir^n (contributes to $\partial\omega_x/\partial t$, $\partial\omega_y/\partial t$) because $\nabla p \approx -\rho g \hat{k}$ so $\nabla\rho \times \nabla p$ is mostly $\perp \hat{k}$.

Let's see how baroclinic term works. Suppose density varies in horizontal. Consider 2 neighboring parcels on an isobar. Consider pgf per unit volume acting on these parcels.



Analogous to kicking heavy and light soccer balls w/ equal force (soccer balls represent air parcels). Light ball accelerates more. [$\vec{F} = m\vec{a}$ $\therefore$ $\vec{a} = \vec{F}/m$ so for fixed $\vec{F}$, bigger m means smaller $\vec{a}$]



$\therefore$ tendency to generate vort comp into page

Baroclinic effect generates horizizontal vorticity at leading edge of cold outflows (gust fronts), cold-fronts, and on edges of clouds and t-storms.