METR 5113, Advanced Atmospheric Dynamics I
Alan Shapiro, Instructor
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1 handout: Problem set 1

An important 2nd order tensor is the "unit tensor" δ (also known as "Kronecker delta tensor" or "substitution tensor"), defined by:

$$\delta_{ij} = 1 \qquad \text{if } i = j$$
$$\quad = 0 \qquad \text{if } i \neq j$$

or equivalently: $\delta_{ij} = \hat{e}_i \cdot \hat{e}_j$ or equivalently: $\delta_{ij} = \dfrac{\partial x_i}{\partial x_j}$

or equivalently (matrix representation): $\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In new (rotated) coords, $\delta'_{ij} = 1 \quad \text{if } i = j$
$\quad = 0 \quad \text{if } i \neq j$

or equivalently $\delta'_{ij} = \hat{e}'_i \cdot \hat{e}'_j$ or equivalently: $\delta'_{ij} = \dfrac{\partial x'_i}{\partial x'_j}$

Show that δ is a 2nd order tensor:

$$\delta'_{mn} = \hat{e}'_m \cdot \hat{e}'_n = \left[\left(\hat{e}'_m \cdot \hat{e}_i\right)\hat{e}_i\right] \cdot \left[\left(\hat{e}'_n \cdot \hat{e}_j\right)\hat{e}_j\right]$$

$$= C_{im}\hat{e}_i \cdot C_{jn}\hat{e}_j = C_{im}C_{jn}\hat{e}_i \cdot \hat{e}_j = C_{im}C_{jn}\delta_{ij} \qquad \text{yep!}$$

δ has a <u>substitution property</u>:

$$\boxed{\delta_{ij}}\, u_j = u_i$$

$\downarrow$

0 <u>except</u> when j = i. Since i is either 1, 2 <u>or</u> 3, while j is <u>summed</u> over 1, 2, <u>and</u> 3, j eventually "hits" i and this yields the only survivor.

for i = 2: $\delta_{2j} u_j = \underset{0}{\delta_{21}} u_1 + \underset{1}{\delta_{22}} u_2 + \underset{0}{\delta_{23}} u_3 = u_2$

Similarly, $\delta_{ij} M_{kpqrsj} = M_{kpqrsi}$

From its definition, the comps of δ are the same in every coord system ($\delta'_{ij} = \delta_{ij}$) so it's an <u>isotropic</u> <u>tensor</u>. δ is the <u>only</u> 2nd order isotropic tensor (apart from const times δ).

Show that the gradient of a scalar F is a vector:

$$\vec{M} \equiv \nabla F = \hat{e}_1 \frac{\partial F}{\partial x_1} + \hat{e}_2 \frac{\partial F}{\partial x_2} + \hat{e}_3 \frac{\partial F}{\partial x_3} = \hat{e}_k \frac{\partial F}{\partial x_k}$$

In original system, comp^s of $\vec{M}$ are: $M_i = \dfrac{\partial F}{\partial x_i}$

In new system, comp^s of $\vec{M}$ are: $M'_j = \dfrac{\partial F}{\partial x'_j}$

so $M'_j = \dfrac{\partial F}{\partial x'_j} = \dfrac{\partial \boxed{x_i}}{\partial x'_j} \dfrac{\partial F}{\partial x_i}$ use fact that $x_i = C_{im} x'_m$

$$= C_{im} \frac{\partial x'_m}{\partial x'_j} \frac{\partial F}{\partial x_i} = C_{im} \delta'_{mj} \frac{\partial F}{\partial x_i} = C_{im} \delta_{mj} \frac{\partial F}{\partial x_i}$$

$$= C_{ij} \frac{\partial F}{\partial x_i} \qquad \text{(using substitution prinicple)}$$

$$= C_{ij} M_i$$

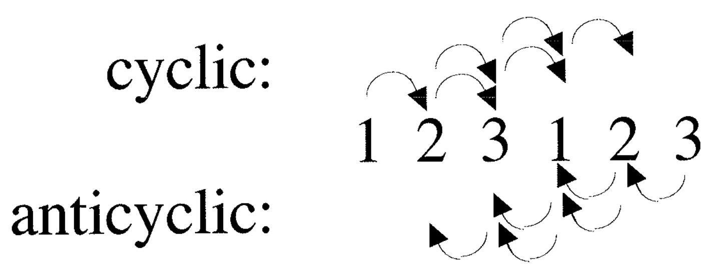
$\therefore \quad \vec{M}$ is a vector.

An important 3rd order tensor is the "epsilon tensor" ε (also known as "alternating tensor"), which has components:

$\varepsilon_{ijk} = 1$ if i, j, k are in cyclic order 1,2,3 or 2,3,1 or 3,1,2

$= 0$ if any 2 indices are equal 1, 1, 3 or 3, 3, 2 etc

$= -1$ if i, j, k are in anticyclic order 3,2,1 or 1,3,2 or 2,1,3



An alternate definition of epsilon: $\varepsilon_{ijk} = \hat{e}_i \cdot \left(\hat{e}_j \times \hat{e}_k\right)$ [keep in mind these basis vectors form a right-handed triple].

With this alternate definition, it's easy to show that ε satisfies the transformation law for 3rd order tensors (try it), i.e., that

$$\varepsilon'_{mnp} = C_{im} C_{jn} C_{kp} \varepsilon_{ijk} .$$

A <u>very</u> important tensor identity is:

$$\boxed{\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}}$$ <u>epsilon-delta (ε-δ) identity</u>

Can verify this identity by brute force, by plugging in all values for indices. Here are some of the steps:

First look at $i = 1, j = 1$, with l, m anything

$$\varepsilon_{11k}\varepsilon_{klm} \overset{?}{=} \delta_{1l}\delta_{1m} - \delta_{1m}\delta_{1l}$$

0 terms on rhs cancel -- get 0

So $0 = 0$ [yep, that works]

Same thing happens if $i = 2, j = 2$ with l, m = anything
and if $i = 3, j = 3$ with l, m = anything

Next look at $i = 1, j = 2$, with l, m = anything.

$$\varepsilon_{12k}\varepsilon_{klm} \overset{?}{=} \delta_{1l}\delta_{2m} - \delta_{1m}\delta_{2l}$$

only survivor on lhs is when $k = 3$ ($\varepsilon_{123} = 1$)

So does $\varepsilon_{3lm} \overset{?}{=} \delta_{1l}\delta_{2m} - \delta_{1m}\delta_{2l}$

for $l = 1$, m = 2:

$$\varepsilon_{312} \qquad \delta_{11}\delta_{22} - \delta_{12}\delta_{21}$$

1   1 1 0 0

So $1 = 1$ [good]

for $l = 2$, m = 1:

$$\varepsilon_{321} \qquad \delta_{12}\delta_{21} - \delta_{11}\delta_{22}$$

-1   0 0 1 1

So $-1 = -1$ [good]

All other l,m combinations have at least one of l or m being 3 so that $\varepsilon_{3lm} = 0$. Can show rhs is also 0.

ETC. ____

Another important property of ε:

$$\varepsilon_{ijk} = -\varepsilon_{jik} ,$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

i.e., interchanging adjacent indices changes sign (changes cyclic order into anticyclic order and vice versa) [adjacent in 123123].

So, interchanging adjacent indices twice gets back the original:

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = -\left(-\varepsilon_{kij}\right) = \varepsilon_{kij}$$

Equivalently, moving 1 index 2 places gives back the original quantity.

<u>Cross product between 2 vectors</u>:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{e}_1\left(u_2 v_3 - u_3 v_2\right) + \hat{e}_2\left(u_3 v_1 - u_1 v_3\right) + \hat{e}_3\left(u_1 v_2 - u_2 v_1\right)$$

$$= \varepsilon_{ijk}\, u_i\, v_j\, \hat{e}_k$$

So,

$$\boxed{\vec{u} \times \vec{v} = \varepsilon_{ijk} u_i v_j \hat{e}_k}$$

or, equivalently:

$$\boxed{\vec{u} \times \vec{v} = \varepsilon_{kij} u_i v_j \hat{e}_k}$$

$\vec{u} \times \vec{v}$ is a vector [can prove it but it's not easy].

What is the "m" component of $\vec{u} \times \vec{v}$? By inspection:

$$\left(\vec{u} \times \vec{v}\right)_m = \varepsilon_{mij} u_i v_j$$

or show it directly:

$$\left(\vec{u} \times \vec{v}\right)_m = \left(\vec{u} \times \vec{v}\right) \cdot \hat{e}_m = \varepsilon_{kij} u_i v_j \hat{e}_k \cdot \hat{e}_m$$

$$= \varepsilon_{kij} u_i v_j \delta_{mk} \qquad \text{from definition of } \delta$$

$$= \varepsilon_{mij} u_i v_j \qquad \text{from substitution principle}$$

<u>Curl of a vector:</u>

$$\nabla \times \vec{u} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= \hat{e}_1\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \hat{e}_2\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \hat{e}_3\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

$$= \varepsilon_{kij}\frac{\partial u_j}{\partial x_i}\hat{e}_k$$

<u>Gradient (del) operator:</u>

$$\nabla(\ ) = \hat{e}_i\frac{\partial(\ )}{\partial x_i}$$

<u>Divergence of a vector:</u>

$$\nabla \cdot \vec{u} = \hat{e}_i\frac{\partial}{\partial x_i} \cdot \hat{e}_j u_j = \hat{e}_i \cdot \hat{e}_j\frac{\partial u_j}{\partial x_i}$$

$$= \delta_{ij}\frac{\partial u_j}{\partial x_i} \quad \text{(now use substitution principle)}$$

$$= \frac{\partial u_i}{\partial x_i} \quad \left(\text{or } \frac{\partial u_j}{\partial x_j} \text{ or ...}\right)$$