METR 5113, Advanced Atmospheric Dynamics I
Alan Shapiro, Instructor
Wednesday, 31 Rocktober 2018 (lecture 30)

- 1 handout: vort tilting in a thunderstorm updraft

Vorticity eq^n (with all terms except local deriv put on rhs):

$$\frac{\partial \vec{\omega}}{\partial t} = -(\vec{u} \cdot \nabla)\,\vec{\omega} + (\vec{\omega} \cdot \nabla)\,\vec{u} + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nu\,\nabla^2\vec{\omega}$$

Vertical vorticity eq^n

Take $\hat{k}\,\cdot$ vort eq^n, get:

$$\frac{\partial \zeta}{\partial t} = -(\vec{u} \cdot \nabla)\,\zeta + (\vec{\omega} \cdot \nabla)\,w + \frac{1}{\rho^2}\,\hat{k} \cdot \left(\nabla\rho \times \nabla p\right) + \nu\,\nabla^2\zeta$$

Each term on rhs contributes to $\partial\zeta/\partial t$. Examine contributions separately.

Discuss vorticity advection term $-(\vec{u} \cdot \nabla)\,\zeta$. [done on board]

Now look at tilting and stretching term $(\vec{\omega} \cdot \nabla)\,w$.

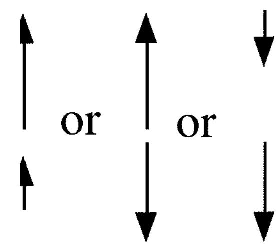
$$\left(\vec{\omega} \cdot \nabla\right) w = \underbrace{\boxed{\omega_x \frac{\partial w}{\partial x} + \omega_y \frac{\partial w}{\partial y}}}_{\text{tilting terms}} + \underbrace{\boxed{\zeta \frac{\partial w}{\partial z}}}_{\text{stretching}}$$

stretching term: $\zeta\,\partial w/\partial z$.

Suppose $\partial w/\partial z > 0$ and $\zeta > 0$ $\therefore$ $\zeta\,\partial w/\partial z > 0$ $\therefore$ $\partial\zeta/\partial t > 0$ $\therefore$ $\zeta \uparrow$ in value. And since $\zeta > 0$, ζ increases in magnitude.

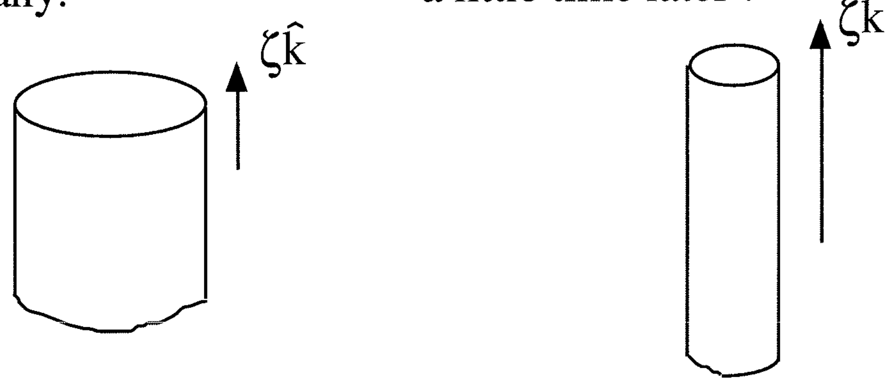
If $\partial w/\partial z > 0$ but $\zeta < 0$ then $\zeta\, \partial w/\partial z < 0$ $\therefore$ $\partial\zeta/\partial t < 0$ $\therefore$ $\zeta \downarrow$ in value. And since $\zeta < 0$, ζ still increases in magnitude.

Recall that linear strain rate in z direction, $\frac{1}{\delta z}\frac{D\delta z}{Dt}$, is related to flow by: $\frac{1}{\delta z}\frac{D\delta z}{Dt} = \frac{\partial w}{\partial z}$. So positive $\frac{\partial w}{\partial z}$ stretches an air parcel. $\partial w/\partial z > 0$ means that w might look like:



So:

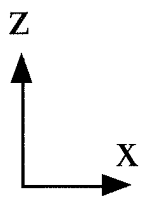
initially: a little time later :



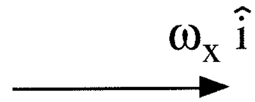
As parcel stretches, its mass is brought in toward axis of rotation, and spins faster ($\zeta \uparrow$ in magnitude).

In calculation above we showed stretching ($\partial w/\partial z > 0$) increases magnitude of ζ <u>regardless of sign of</u> ζ. Similarly, you can show compression ($\partial w/\partial z<0$) reduces mag of ζ <u>regardless of sign of</u> ζ.

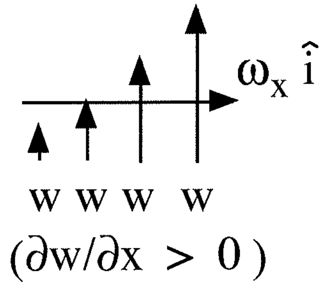
<u>one of the tilting terms:</u> $\omega_x\, \partial w/\partial x$.
Suppose $\omega_x > 0$ and $\partial w/\partial x > 0$. Then $\omega_x\, \partial w/\partial x > 0$. $\therefore$ $\partial\zeta/\partial t > 0$. So $\zeta \uparrow$ in value.



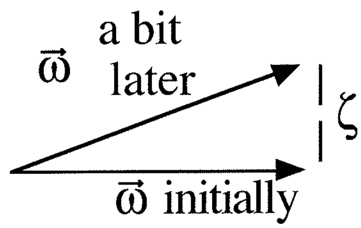
Consider horiz vortex line directed in x dir^n



Impose an updraft increasing in x dir^n



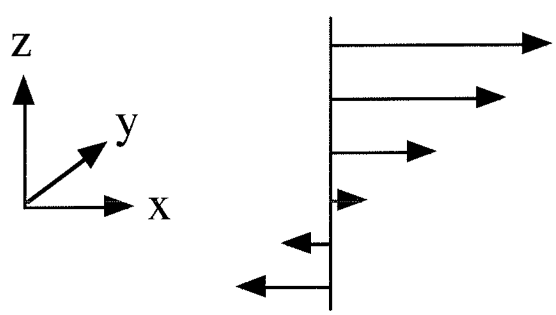
w field rotates initially horiz vortex line, generates positive ζ:



horiz vorticity gets "tilted" into vertical dir^n. A rearrangement of vorticity.

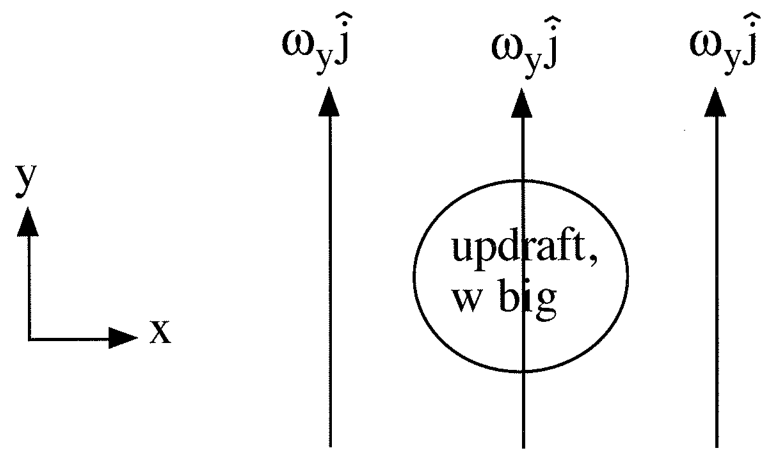
Similar interpretation for $\omega_y \, \partial w/\partial y$ tilting term.

e.g., generation of vertical vorticity in thunderstorms grown in a sheared environment. Let the environmental (mean) winds be: $\bar{u} = \bar{u}(z), \quad \bar{v} = 0, \quad \bar{w} = 0$.



∴ environmental (mean) vorticity is in y dir^n

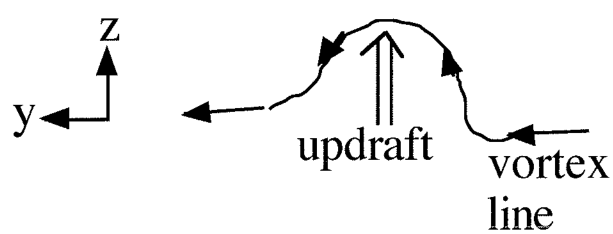
Now consider an updraft growing in that environment.
In plan view:



environmental vortex lines

South of updraft center, vortex lines get tilted upward ∴ pos ζ generated. North of updraft center get neg ζ generated.

Looking to the east at the updraft:



The tilting is consistent with Helmholtz thm.

Note: in 2D flow, tilting and stretching mechanisms don't operate. Why? If u =u(x, y), v = v(x, y), w=0, then vort is all in vertical: $\vec{\omega} = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\hat{k}$ ∴ no horiz vort available for tilting into vertical. Also, since w =0, $\frac{\partial w}{\partial z} = 0$ so $\zeta \frac{\partial w}{\partial z} = 0$ so no stretching. Similarly, can show that tilting/stretching terms are also 0 in the x and y component vorticity equations.

Ertel's Potential Vorticity Theorem

Assume flow is inviscid, but let it be baroclinic and compressible. Mass cons^n eq^n for compressible flow is:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

Eq^{ns} of motion are Euler eq^{ns} (since flow is inviscid):

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\,\vec{u} = -\frac{1}{\rho}\nabla p - \nabla \Pi$$

Taking $\nabla \times$ Euler eq^{ns} and using the identities for $\nabla \times (\vec{\omega} \times \vec{u})$ and $\nabla \cdot \vec{\omega}$, we get:

$$\frac{\partial \vec{\omega}}{\partial t} + \left(\vec{u} \cdot \nabla\right)\vec{\omega} + \vec{\omega}\,\boxed{\nabla \cdot \vec{u}} = \left(\vec{\omega} \cdot \nabla\right)\vec{u} + \frac{1}{\rho^2}\nabla \rho \times \nabla p$$

$\downarrow$ not 0!

$$-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho\right)$$

$\div$ by ρ and re-arrange:

(A) $$\frac{1}{\rho}\frac{\partial \vec{\omega}}{\partial t} - \frac{\vec{\omega}}{\rho^2}\frac{\partial \rho}{\partial t} + \frac{1}{\rho}\left(\vec{u} \cdot \nabla\right)\vec{\omega} - \frac{\vec{\omega}}{\rho^2}\vec{u} \cdot \nabla \rho$$

$$= \left(\frac{\vec{\omega}}{\rho} \cdot \nabla\right)\vec{u} + \frac{1}{\rho^3}\nabla \rho \times \nabla p$$

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scratch paper:

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{\partial}{\partial t}\rho^{-1} = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial t}$$ Similarly, $$\nabla\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\nabla \rho$$

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so lhs of (A) can be rewritten as:

$$\left(\frac{1}{\rho}\right)\frac{\partial\vec{\omega}}{\partial t} + \vec{\omega}\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}\right)(\vec{u}\cdot\nabla)\vec{\omega} + \vec{\omega}\,(\vec{u}\cdot\nabla)\left(\frac{1}{\rho}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\vec{\omega}}{\rho}\right) + (\vec{u}\cdot\nabla)\left(\frac{\vec{\omega}}{\rho}\right)$$

$$= \frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right)$$

So (A) becomes:

$$\text{(B)} \qquad \boxed{\frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = \left(\frac{\vec{\omega}}{\rho}\cdot\nabla\right)\vec{u} + \frac{1}{\rho^3}\nabla\rho\times\nabla p}$$