METR 5113, Advanced Atmospheric Dynamics I
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Ertel's Potential Vorticity Theorem (continued)

The vorticity eqn for an inviscid compressible and baroclinic flow can be written as

$$\text{(B)} \qquad \frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = \left(\frac{\vec{\omega}}{\rho}\cdot\nabla\right)\vec{u} + \frac{1}{\rho^3}\nabla\rho\times\nabla p$$

Now suppose there is some scalar quantity λ that is conserved:

$$\frac{D}{Dt}\lambda = 0 \quad \rightarrow \quad \lambda = \text{const (for a parcel)}$$

λ is a "passive scalar" or "marker" for the flow, e.g., can choose:

$\lambda = \theta$ (pot temp) if flow is isentropic,

$\lambda =$ ozone concentration (for short time scales),

$\lambda =$ concentration of a non-reactive pollutant.

Even if λ is conserved, does not mean that $\nabla\lambda$ is conserved! [--> | | | | <-- becomes ||||. Gradient can also rotate, e.g., in presence of a shear]

Look at $\frac{D}{Dt}\nabla\lambda$ (well, one component of it):

$$\frac{D}{Dt}\frac{\partial\lambda}{\partial x_i} = \frac{\partial}{\partial t}\left(\frac{\partial\lambda}{\partial x_i}\right) + u_j\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda}{\partial x_i}\right)$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial \lambda}{\partial t}\right) + u_j \frac{\partial}{\partial x_i}\left(\frac{\partial \lambda}{\partial x_j}\right)$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial \lambda}{\partial t}\right) + \frac{\partial}{\partial x_i}\left(u_j \frac{\partial \lambda}{\partial x_j}\right) - \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial \lambda}{\partial t} + u_j \frac{\partial \lambda}{\partial x_j}\right) - \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{D\lambda}{Dt}\right) - \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j}$$ but $\frac{D\lambda}{Dt} = 0$ since λ is conserved

$$\therefore \quad \frac{D}{Dt}\frac{\partial \lambda}{\partial x_i} = -\frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j}$$ contract with ω_i:

$$\omega_i \frac{D}{Dt}\frac{\partial \lambda}{\partial x_i} = -\omega_i \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j}$$

$$\therefore \quad \vec{\omega} \cdot \frac{D}{Dt}\nabla\lambda = -\nabla\lambda \cdot \left(\vec{\omega} \cdot \nabla\right)\vec{u}$$ $\div$ by ρ:

(C) $$\boxed{\frac{\vec{\omega}}{\rho} \cdot \frac{D}{Dt}\nabla\lambda = -\nabla\lambda \cdot \left(\frac{\vec{\omega}}{\rho} \cdot \nabla\right)\vec{u}}$$

Take $\nabla\lambda \cdot (B)$

$$\nabla\lambda \cdot \frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = \nabla\lambda \cdot \left(\frac{\vec{\omega}}{\rho} \cdot \nabla\right)\vec{u} + \frac{1}{\rho^3}\nabla\lambda \cdot \left(\nabla\rho \times \nabla p\right)$$

Add this eq^n to (C), get:

$$\frac{\vec{\omega}}{\rho} \cdot \frac{D}{Dt} \nabla\lambda + \nabla\lambda \cdot \frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = \frac{1}{\rho^3} \nabla\lambda \cdot \left(\nabla\rho \times \nabla p\right)$$

$$\frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho} \cdot \nabla\lambda\right) = \frac{1}{\rho^3} \boxed{\nabla\lambda \cdot \left(\nabla\rho \times \nabla p\right)}$$

scalar triple product

rhs is 0 if:

(i) density is constant

or (ii) flow is barotropic

or (iii) $\lambda = \lambda(p)$ or $\lambda = \lambda(\rho)$

or (iv) $\lambda = \lambda(p, \rho)$

[Prove it. Need chain rule for (iii) and (iv).]

Assuming one of these latter conditions is true, we get:

$$\boxed{\frac{D}{Dt}\left(\frac{\vec{\omega} \cdot \nabla\lambda}{\rho}\right) = 0}$$ <u>Ertel's Potential Vorticity Theorem</u>

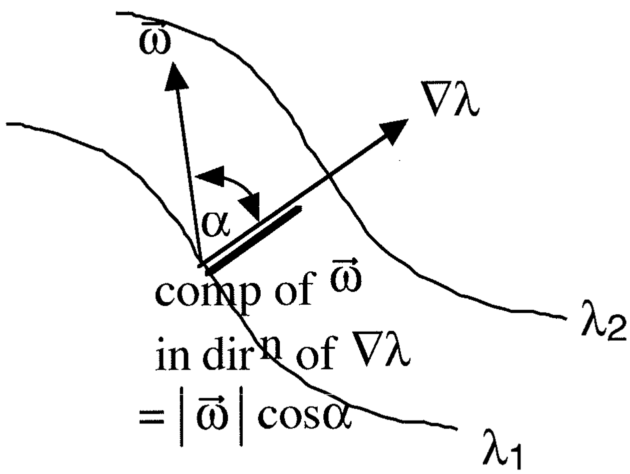
or,

$$\boxed{\frac{\vec{\omega} \cdot \nabla\lambda}{\rho} = \text{const}}$$ for a parcel (dif. consts for dif parcels).

$\dfrac{\vec{\omega} \cdot \nabla\lambda}{\rho}$ is Ertel potential vorticity

$$= \frac{|\nabla\lambda| \, \boxed{|\vec{\omega}| \cos\alpha}}{\rho} \quad \text{where } \alpha \text{ is angle btw } \vec{\omega} \text{ and } \nabla\lambda .$$

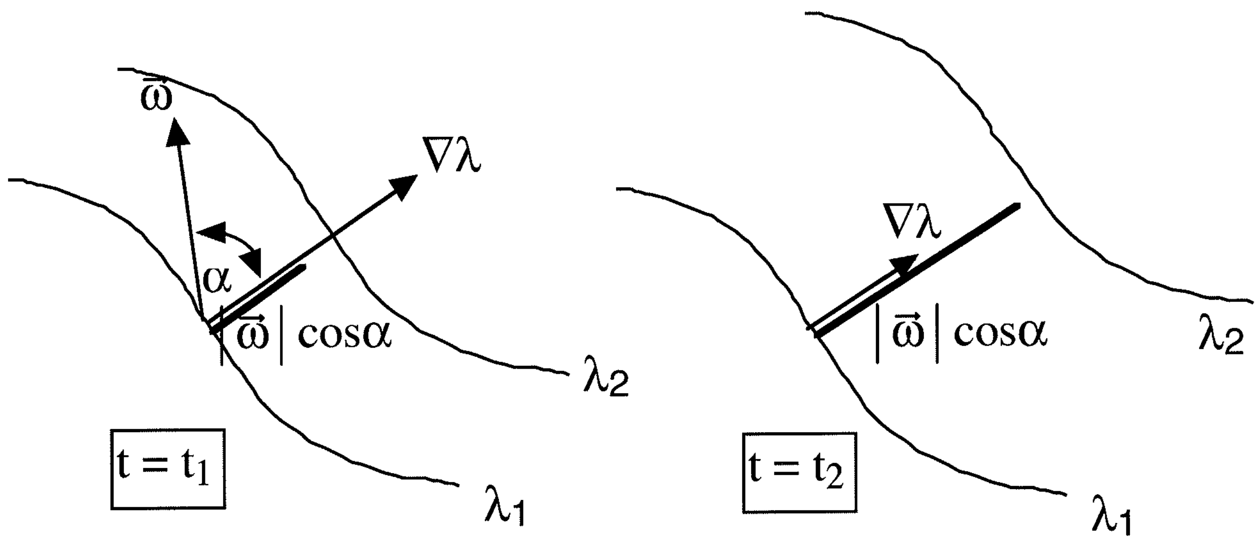
Product of <u>magnitude of gradient of conserved scalar</u> and <u>comp of vorticity in direction of that gradient</u> ($\div \rho$) is const for a parcel.



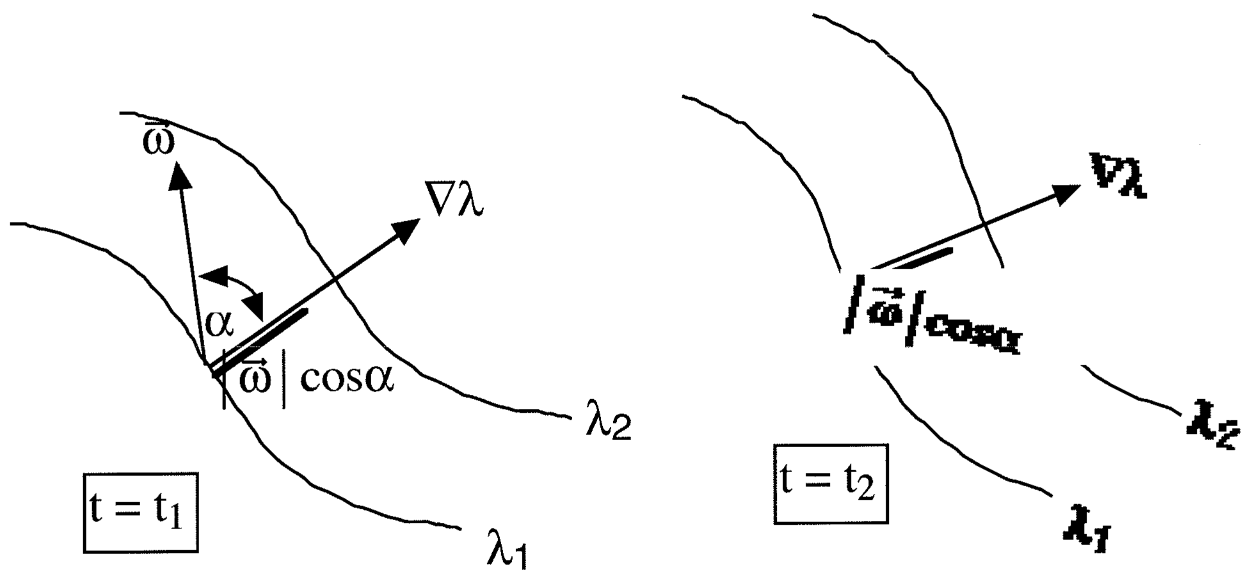
[Comp of $\vec{\omega} \perp \nabla\lambda$ is irrelevant to thm.]

e.g. to isolate role of ρ, consider case where parcel moves downward $\therefore$ parcel compresses, so $\rho \uparrow$ so Ertel's thm says vorticity $\uparrow$ (provided we ignore changes in $|\nabla\lambda|$). What happened? Mass was brought in closer to rot axis so vort $\uparrow$.

e.g., suppose flow pulls λ sfcs apart. In that case $|\nabla\lambda| \downarrow$ $\therefore$ comp of $\vec{\omega}$ in dir^n of $\nabla\lambda \uparrow$. Increase is due to <u>stretching</u>:



e.g., suppose flow rotates λ sfcs w/out changing magnitude of $\nabla\lambda$:



- so flow rotates $\nabla\lambda$ vector w/out changing magnitude of $\nabla\lambda$ so <u>magnitude</u> of comp of $\vec{\omega}$ in dir^n of $\nabla\lambda$ stays the same. [in this example vort vector would also rotate w/ flow]