METR 5113, Advanced Atmospheric Dynamics I
Alan Shapiro, Instructor
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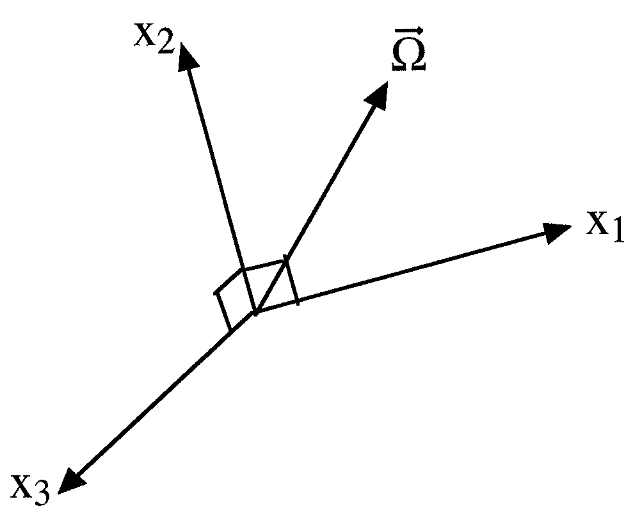
Eqns of motion in a rotating reference frame [Kundu Ch 4]

Reference frame refers to motion of a hypothetical observer. If observer is not accelerating the ref frame is non-accelerating (a.k.a. an inertial or fixed ref frame). If observer is accelerating, the ref frame is accelerating (a.k.a. a non-inertial ref frame).

Newton's 2nd law and eqns derived from it (N-S eqns, vort eqn, etc) were considered in an inertial reference frame. If we want to use them in a non-inertial ref frame we must modify them.

If a ref frame is ... not moving, then it's an inertial ref frame.
... moving w/ constant velocity (const speed and direction) then it's an inertial ref frame.
... rotating with constant angular velocity then it's a non-inertial reference frame.

Consider a frame of ref rotating with constant angular velocity $\vec{\Omega}$ (say, earth ang velocity). Work with a Cartesian coord system (x_1, x_2, x_3) w/ unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ with origin on rotation axis.



This Cartesian coord system will be used to help us derive the vector form of the eq^{ns} of motion in the rotating ref frame. The vector form of these eq^{ns} cares about the ref frame but doesn't care about the coord system. [Use Cart coords to most easily get at the vector form. The vector eq^{ns} can then be decomposed into any coord system (Cartesian, cylindrical, spherical, etc)]

Can write any vector $\vec{P}$ in the rotating Cartesian system as:

$$\vec{P} = P_1\,\hat{e}_1 + P_2\,\hat{e}_2 + P_3\,\hat{e}_3 = P_j\,\hat{e}_j$$

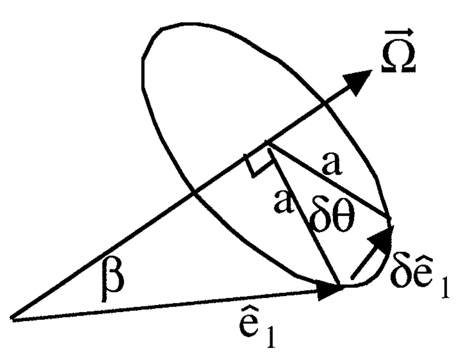
The components P_j are what are observed in the rot ref frame.

To an observer in an inertial ref frame, the unit vectors of the Cartesian coord system are observed to spin around the rotation axis, i.e., they change with time. So total deriv of $\vec{P}$ is:

$$\left(\frac{d\vec{P}}{dt}\right)_F = \frac{dP_j}{dt}\hat{e}_j + P_j\frac{d\hat{e}_j}{dt} = \left(\frac{d\vec{P}}{dt}\right)_R + P_j\frac{d\hat{e}_j}{dt}$$

F = obs in fixed frame, R = obs in rotating frame

Can relate $\frac{d\hat{e}_j}{dt}$ to $\vec{\Omega}$. Look at $\hat{e}_1$. In a fixed frame, tip of $\hat{e}_1$ traces a circle around rotation axis:



Find $\frac{d\hat{e}_1}{dt}$ $(= \lim_{\delta t \to 0} \frac{\delta \hat{e}_1}{\delta t})$ using fact that "<u>any vector can be written as product of its magnitude and unit vector in its direction</u>"

First let's find the magnitude, $\left|\frac{d\hat{e}_1}{dt}\right|$. The change in $\hat{e}_1$ over a small time interval δt is $\delta\hat{e}_1$. The magnitude of this change is:

$$\left|\delta\hat{e}_1\right| = a\,\delta\theta$$

$$\sin\beta = a/\left|\hat{e}_1\right| = a$$

$$\therefore\ \left|\delta\hat{e}_1\right| = \sin\beta\,\delta\theta$$

$$\therefore\ \left|\frac{d\hat{e}_1}{dt}\right| = \lim_{\delta t \to 0} \frac{\left|\delta\hat{e}_1\right|}{\delta t} = \lim_{\delta t \to 0} \sin\beta \frac{\delta\theta}{\delta t} = \sin\beta \frac{d\theta}{dt}$$

$$= \sin\beta\left|\vec{\Omega}\right| = \left|\vec{\Omega}\times\hat{e}_1\right|.$$ There, that's the magnitude...

From previous diagram we see <u>direction</u> of $d\hat{e}_1/dt$ (dir^n of $\delta\hat{e}_1$) is $\perp$ to both $\hat{e}_1$ and $\vec{\Omega}$. Specifically, $d\hat{e}_1/dt$ points in dir^n of $\vec{\Omega}\times\hat{e}_1$. The unit vector in that direction is $\frac{\vec{\Omega}\times\hat{e}_1}{\left|\vec{\Omega}\times\hat{e}_1\right|}$.

Put 'em together: $$\frac{d\hat{e}_1}{dt} = \underbrace{\left|\vec{\Omega}\times\hat{e}_1\right|}_{\text{magnitude}} \underbrace{\frac{\vec{\Omega}\times\hat{e}_1}{\left|\vec{\Omega}\times\hat{e}_1\right|}}_{\text{unit vector}} = \vec{\Omega}\times\hat{e}_1$$

Get similar results for other unit vectors. So:

$$\therefore \left(\frac{d\vec{P}}{dt}\right)_F = \left(\frac{d\vec{P}}{dt}\right)_R + \boxed{P_j \vec{\Omega} \times \hat{e}_j} \longrightarrow \boxed{\vec{\Omega} \times P_j \hat{e}_j} \longrightarrow \boxed{\vec{\Omega} \times \vec{P}}$$

$$\therefore \boxed{\left(\frac{d\vec{P}}{dt}\right)_F = \left(\frac{d\vec{P}}{dt}\right)_R + \vec{\Omega} \times \vec{P}}$$

Let $\vec{P}$ be the pos^n vector $\vec{r}$ of an air parcel:

$$\left(\frac{d\vec{r}}{dt}\right)_F = \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{r}$$

$$\therefore \vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

absolute velocity — relative velocity — velocity of ref frame at location $\vec{r}$ [solid body rotation velocity at location $\vec{r}$]

[at 30° lat, relative velocity is ~10 mph but solid body rot velocity is ~850 mph]

Now let $\vec{P} = \vec{u}_F$

$$\left(\frac{d\vec{u}_F}{dt}\right)_F = \left(\frac{d\vec{u}_F}{dt}\right)_R + \vec{\Omega} \times \vec{u}_F \quad \text{(use formula for } \vec{u}_F \text{ for rhs)}$$

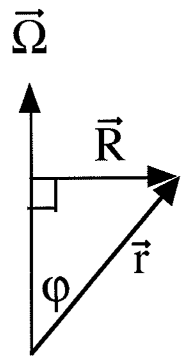
$$= \left(\frac{d}{dt}\left(\vec{u}_R + \vec{\Omega} \times \vec{r}\right)\right)_R + \vec{\Omega} \times \left(\vec{u}_R + \vec{\Omega} \times \vec{r}\right)$$

$$= \left(\frac{d\vec{u}_R}{dt}\right)_R + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$

$$\therefore \left(\frac{d\vec{u}_F}{dt}\right)_F = \left(\frac{d\vec{u}_R}{dt}\right)_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$

acceleration obs in fixed ref frame — acceleration obs in rot ref frame

Write last term in terms of a pos^n vector $\vec{R}$ drawn $\perp$ to rot axis:



$$|\vec{R}| = |\vec{r}|\sin\varphi$$

$$\vec{\Omega}\times\vec{r} = |\vec{\Omega}||\vec{r}|\sin\varphi\,\boxed{\hat{t}} \longrightarrow \text{unit vector into page} \quad = |\vec{\Omega}||\vec{R}|\,\hat{t}$$

$$\therefore\ \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) = \vec{\Omega}\times|\vec{\Omega}||\vec{R}|\,\hat{t} = |\vec{\Omega}||\vec{R}|\,\vec{\Omega}\times\hat{t}$$

Find $\vec{\Omega}\times\hat{t}$ using "Any vector can be written as product of its magnitude and the unit vector in its direction"

magnitude: $\quad |\vec{\Omega}\times\hat{t}| = |\vec{\Omega}||\hat{t}|\sin 90 = |\vec{\Omega}|$

direction $\quad \vec{\Omega}\times\hat{t}$ points in dir^n of $-\vec{R}$

unit vector in that direction: $\quad -\dfrac{\vec{R}}{|\vec{R}|}$

put 'em together: $\quad \vec{\Omega}\times\hat{t} = -|\vec{\Omega}|\dfrac{\vec{R}}{|\vec{R}|}$

$$\therefore\ \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) = |\vec{\Omega}||\vec{R}|\left(-|\vec{\Omega}|\frac{\vec{R}}{|\vec{R}|}\right) = -|\vec{\Omega}|^2\vec{R} = -\Omega^2\vec{R}$$

where $\Omega \equiv |\vec{\Omega}|$

$$\therefore \quad \frac{D\vec{u}_F}{Dt} = \frac{D\vec{u}_R}{Dt} + 2\vec{\Omega} \times \vec{u}_R - \Omega^2 \vec{R}$$

Navier-Stokes equations in <u>fixed frame of ref</u>:

$$\boxed{\frac{D\vec{u}_F}{Dt}} = -\frac{1}{\rho}\nabla p + \vec{g} + \nu \nabla^2 \boxed{\vec{u}_F}$$

Want to rewrite it in rot ref frame:

$$\frac{D\vec{u}_R}{Dt} + 2\vec{\Omega} \times \vec{u}_R - \Omega^2 \vec{R} = -\frac{1}{\rho}\nabla p + \vec{g} + \nu \nabla^2 \left(\vec{u}_R + \boxed{\vec{\Omega} \times \vec{r}} \right)$$

scratch paper---

$$\nabla^2 (\vec{\Omega} \times \vec{r}) = \vec{\Omega} \times \nabla^2 \vec{r} = \vec{\Omega} \times \nabla^2 \left(x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \right) = 0$$

since $\nabla^2 x = 0, \ \nabla^2 y = 0, \ \nabla^2 z = 0$

Drop "R" subscript (w/ understanding that all quanitities are still obs in rot ref frame) and rearrange, get <u>N-S eqns in Rot Frame:</u>

$$\boxed{\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p \quad \underbrace{-\,2\vec{\Omega} \times \vec{u}}_{\text{Coriolis force}} \quad + \quad \underbrace{\Omega^2 \vec{R}}_{\text{centrifugal force}} \quad + \quad \vec{g} + \nu \nabla^2 \vec{u}}$$