METR 5113, Advanced Atmospheric Dynamics I
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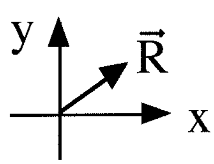
Eq^{ns} of motion (and vort eq^{n}, etc) in rotating ref frame

N-S eqn in Rot Frame is:

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p \quad \underbrace{-2\vec{\Omega}\times\vec{u}}_{\text{Coriolis force}} \quad + \quad \underbrace{\Omega^2\vec{R}}_{\text{centrifugal force}} \quad + \quad \vec{g} + \nu\nabla^2\vec{u}$$

Now lets rewrite $\Omega^2\vec{R}$ as gradient of a scalar potential function.
Orient Cartesian coord system s.t. xy plane contains $\vec{R}$:



$$\vec{R} = x\,\hat{i} + y\,\hat{j}$$

$$R \equiv \left|\vec{R}\right| = \sqrt{x^2 + y^2}$$

$$\therefore \quad R^2 = x^2 + y^2$$

$$\therefore \quad \nabla R^2 = \nabla x^2 + \nabla y^2 = 2x\overset{\hat{i}}{\boxed{\nabla x}} + 2y\overset{\hat{j}}{\boxed{\nabla y}} = 2(x\hat{i} + y\hat{j}) = 2\vec{R}$$

$$\therefore \quad \vec{R} = \nabla R^2/2$$

$$\therefore \quad \Omega^2\vec{R} = \nabla\boxed{\Omega^2 R^2/2} \longrightarrow \text{centrifugal potential}$$

Gravity can be written as gradient of a gravitational potential:

$$\vec{g} = -\nabla\Pi$$

So sum of gravity and centrifugal force can be written as gradient of a potential:

$$\vec{g} + \Omega^2\vec{R} = -\nabla\Pi + \nabla\Omega^2R^2/2 = -\nabla\left(\Pi - \Omega^2R^2/2\right)$$

redefine $\vec{g}$ to be "effective gravity":

$$\underbrace{\vec{g}}_{\text{effective gravity}} = \underbrace{\vec{g}}_{\text{gravity}} + \underbrace{\Omega^2\vec{R}}_{\text{centrifugal force}}$$

$= -\nabla gz$ → z: local height above an equipotential surface; ↓ g: accel due to effective gravity

Note that $\nabla\times\vec{g} = 0$ (for effective $\vec{g}$)

How does vorticity as observed in a fixed (inertial) ref frame differ from vort obs in rotating (non-inertial) ref frame?

$$\underbrace{\vec{\omega}_F}_{\text{abs vort}} \equiv \nabla\times\vec{u}_F = \nabla\times(\vec{u}_R + \vec{\Omega}\times\vec{r})$$

$$= \nabla\times\vec{u}_R + \nabla\times(\vec{\Omega}\times\vec{r})$$

$$= \vec{\omega}_R + \nabla\times(\vec{\Omega}\times\vec{r})$$

$$= \vec{\omega}_R + \vec{\Omega}\underbrace{\boxed{\nabla\cdot\vec{r}}}_{3} + \underbrace{\boxed{(\vec{r}\cdot\nabla)\vec{\Omega}}}_{0} - \underbrace{\boxed{\vec{r}\,\nabla\cdot\vec{\Omega}}}_{0} - (\vec{\Omega}\cdot\nabla)\vec{r}$$

$$= \vec{\omega}_R + 3\vec{\Omega} - \left(\Omega_x \boxed{\frac{\partial \vec{r}}{\partial x}} \to \hat{i} + \Omega_y \boxed{\frac{\partial \vec{r}}{\partial y}} \to \hat{j} + \Omega_z \boxed{\frac{\partial \vec{r}}{\partial z}} \to \hat{k}\right)$$

$$= \vec{\omega}_R + 3\vec{\Omega} - \vec{\Omega}$$

$$\therefore \quad \boxed{\vec{\omega}_F = \vec{\omega}_R + 2\vec{\Omega}}$$

absolute vorticity — relative vorticity — earth vorticity

<u>Now recast other previous equations in rot ref frame:</u>

Can show Bernoulli's eq^n for steady, inviscid, barotropic flow is:

$$\frac{q^2}{2} + \int \frac{dp}{\rho} + gz = C \quad \text{along a streamline}$$

<u>Same eq^n as before</u> but with redefined g, z. q is relative speed (speed obs in rotating ref frame).

Bernoulli's eqn for irrotational, inviscid barotropic flow (possibly unsteady) will also be same as before provided "irrotational" is taken to mean the <u>absolute vorticity</u> is 0.

<u>Circulation th^{ms}</u>

In fixed frame Kelvin's th^m: $D\Gamma_F / Dt = 0$ for closed material curve in inviscid, barotropic flow.

$$\therefore \quad \Gamma_F = C, \text{ or: } \quad \oint \vec{u}_F \cdot \vec{dl} = C$$

sub in $\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$,

$$\oint \vec{u}_R \cdot \vec{dl} \ + \ \oint (\vec{\Omega} \times \vec{r}) \cdot \vec{dl} \ = \ C$$

Γ_R, relative circulation + circulation associated w/ solid body rot. = C

Apply Stokes thm to 2nd term.

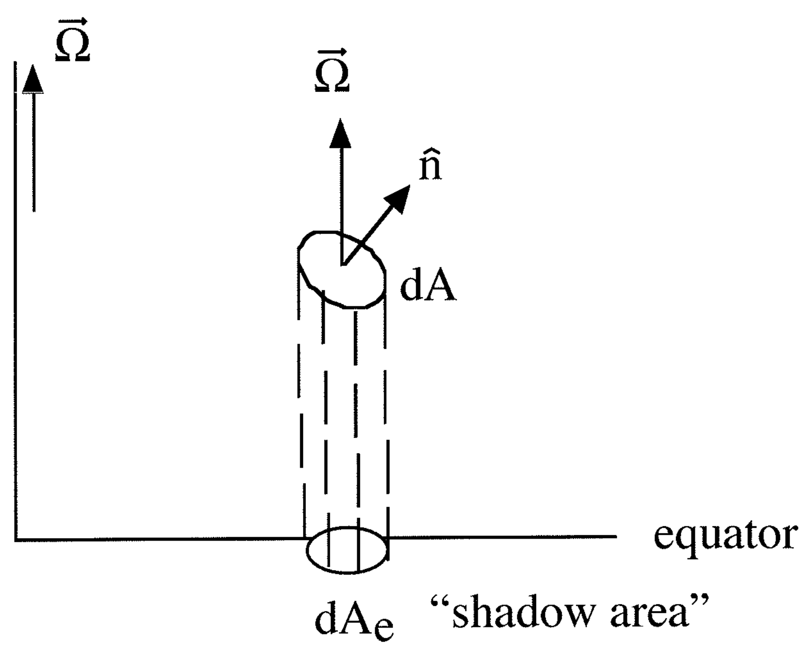
$$\Gamma_R + \int \boxed{[\nabla \times (\vec{\Omega} \times \vec{r})]} \cdot \hat{n}\, dA = C$$

$\downarrow$

$2\vec{\Omega}$ (already showed it)

$$\Gamma_R + \int 2\vec{\Omega} \cdot \hat{n}\, dA = C$$

Rewrite integral in terms of "projected area" or "shadow area." Consider tiny directed area element $\hat{n}\, dA$. Let dA_e be the <u>projection</u> of $\hat{n}\, dA$ onto the equatorial plane (plane $\perp$ to $\vec{\Omega}$).



$$dA_e \equiv \underbrace{\left(\frac{\vec{\Omega}}{\Omega}\right)}_{\text{unit vector}} \cdot \hat{n}\, dA$$

mult by Ω

$\Omega\, dA_e = \vec{\Omega} \cdot \hat{n}\, dA$ [rhs is in Kelvin's thm]

So Kelvin's thm in rot frame is: $\Gamma_R + \int 2\Omega dA_e = C$, or

$$\boxed{\Gamma_R + 2\Omega A_e = C} \quad \text{or} \quad \boxed{\frac{D\Gamma_R}{Dt} + 2\Omega\frac{DA_e}{Dt} = 0}$$

relative circ + earth vort x total projected area is conserved following motion of a material curve.

If flow is <u>baroclinic</u> then circ thm in inertial frame becomes,

$$\frac{D\Gamma_F}{Dt} = -\oint \frac{dp}{\rho}$$

and in rot frame is:

$$\boxed{\frac{D\Gamma_R}{Dt} + 2\Omega\frac{DA_e}{Dt} = -\oint \frac{dp}{\rho}} \quad \text{Bjerknes Circulation Thm}$$

<u>Helmholtz Thm in rotating ref frame</u>:

Absolute vortex lines (lines that are everywhere tangent to <u>absolute</u> vort vectors) move w/ the flow.

Vorticity eqn in rotating ref frame:

$$\frac{\partial \vec{\omega}}{\partial t} + \left(\vec{u} \cdot \nabla\right) \vec{\omega} = \left[\left(\vec{\omega} + 2\vec{\Omega}\right) \cdot \nabla\right] \vec{u} + \frac{1}{\rho^2} \nabla\rho \times \nabla p + \nu \nabla^2 \vec{\omega}$$

Ertel's Pot Vort Thm in rot ref frame:

$$\frac{D}{Dt}\left[\frac{(\vec{\omega} + 2\vec{\Omega}) \cdot \nabla\lambda}{\rho}\right] = 0$$