METR 5113, Advanced Atmospheric Dynamics I
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1 handout: Problem set 5

Rossby waves

Rossby waves are large-scale (planetary) waves on the westerlies. Mountains and other disturbances can generate them, but "waviness" is due to latitudinal variation in f.

To bring out essence of the Rossby wave, consider 2-D incomp flow satisfying (3). Since flow is incomp and 2-D, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ and (3) becomes:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = -\beta v$$

Since flow is incomp and 2D, we also know that:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

So rel vort is: $\zeta = -\nabla^2\psi$.

So vort eqn becomes:

$$(4) \quad -\frac{\partial}{\partial t}\left(\nabla^2\psi\right) - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2\psi\right) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2\psi\right) = \beta\frac{\partial \psi}{\partial x}$$

Fortunately this nonlinear pde admits travelling wave solutions.

Try $\psi = A\cos\frac{2\pi}{\lambda}(x - ct)$ where c is phase speed and λ is wavelength. Plug into (4), get:

$$\left(\frac{2\pi}{\lambda}\right)^3 cA\sin\frac{2\pi}{\lambda}(x - ct) + 0 + 0 = -\beta\,\frac{2\pi}{\lambda}\,A\sin\frac{2\pi}{\lambda}(x - ct)$$

$$\therefore \quad \boxed{c = -\beta\left(\frac{\lambda}{2\pi}\right)^2}$$

$c<0$, so <u>wave propagates toward west.</u> Longer waves are faster.

Now consider a wave superimposed on a <u>westerly current</u> of const speed U: $\psi = Uy + A\cos\frac{2\pi}{\lambda}(x - ct)$. Plug into (4), get:

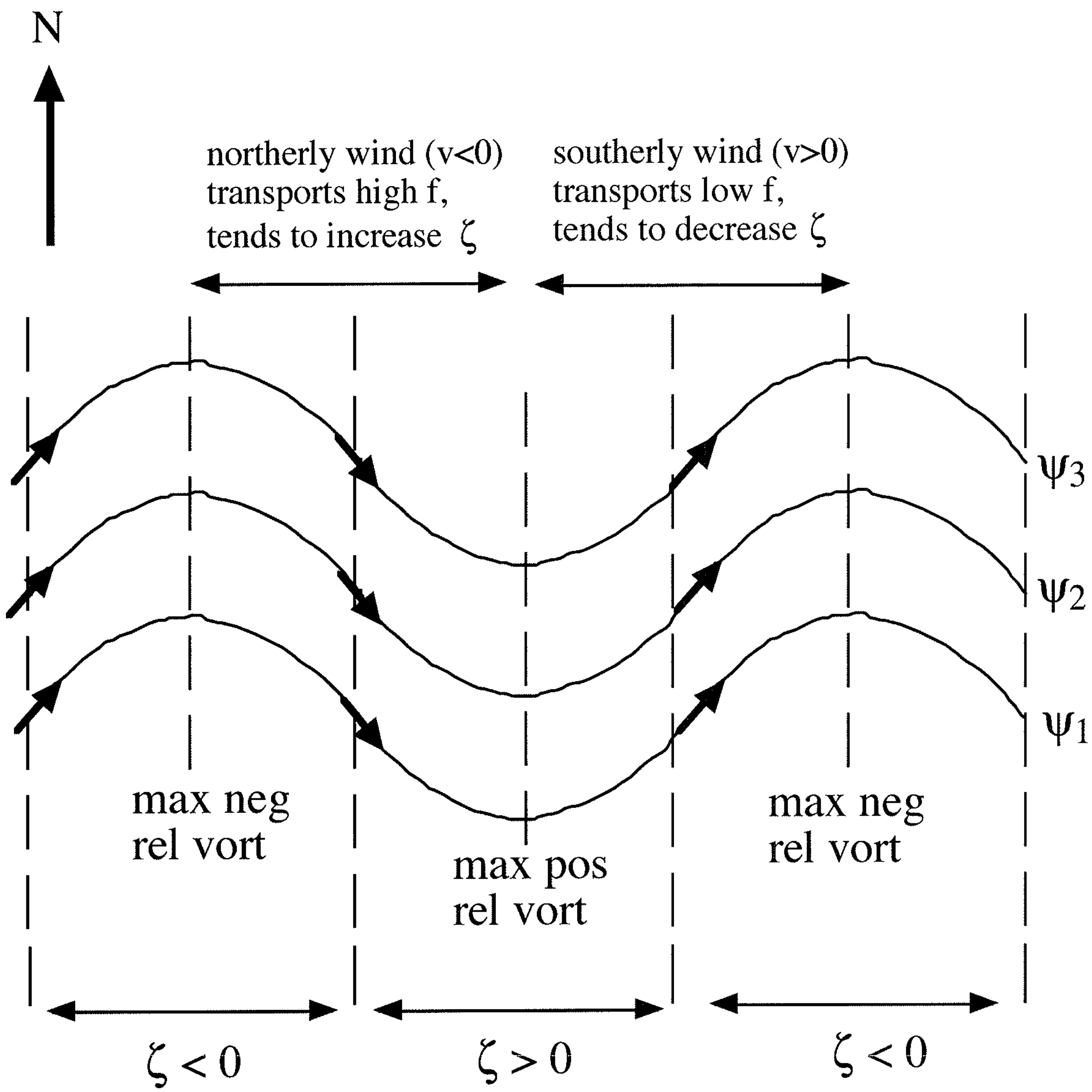
$$\left(\frac{2\pi}{\lambda}\right)^3 c\,A\sin\frac{2\pi}{\lambda}(x - ct) - U\left(\frac{2\pi}{\lambda}\right)^3 A\sin\frac{2\pi}{\lambda}(x - ct) + 0$$

$$= -\beta\,\frac{2\pi}{\lambda}\,A\sin\frac{2\pi}{\lambda}(x - ct)$$

$$\therefore \quad \boxed{c = U - \beta\left(\frac{\lambda}{2\pi}\right)^2}$$

U tries to make wave propagate toward east, while β term (advection of earth vorticity) tries to make wave propagate toward west (retrogress). For short waves , U wins so $c > 0$, so wave propagates eastward. For long waves, $c < 0$, so β term wins and wave retrogresses.

To understand how the wave can retrogress, consider the diagram:



Rewrite the vertical vorticity eqn $\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} = -\beta v$ as $\frac{D}{Dt}(\zeta + f) = 0$. It integrates to: $\zeta + f = \text{const}$ <u>for a parcel</u>. We can use the above diagram together with $\zeta + f = \text{const}$ to explain the "waviness" of the Rossby wave (think curvature vort):

For a parcel with negative ζ moving south, $f \downarrow$ so $\zeta \uparrow$. So then, eventually parcel ζ becomes positive and parcel moves northward (i.e., positive curvature to its trajectory).

Similarly, for a parcel with positive ζ moving north, $f \uparrow$ so $\zeta \downarrow$. So then, eventually parcel ζ becomes negative and parcel turns to move south (negative curvature to its trajectory).

<u>Gravity waves</u> (Kundu, Ch7 of editions 1-5, Ch 8 of 6th edition)

We'll work w/ waves of the form:

some flow property $\sim$ $a \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$ or $a \cos\left[\frac{2\pi}{\lambda}(x - ct)\right]$

a is <u>amplitude</u>

$\frac{2\pi}{\lambda}(x - ct)$ is <u>phase</u> of wave

λ is <u>wavelength</u>

c is <u>phase speed</u>

Why consider waves of this form?

-- Many waves in atmosphere "look" like sines or cosines.

-- Many waves approximately satisfy <u>linear const coeff</u> <u>odes and pdes</u> that permit sin and cos solns. Can use <u>Fourier analysis</u> to get solns of complicated problems by summing sin and cos solns of various amplitudes and wavelengths.

A closer look at wave parameters:

phase: $\frac{2\pi}{\lambda}(x - ct)$

Wave repeats itself when phase changes by 2π. So, at a fixed moment in time, phase changes by 2π when x changes by λ. Hence the name wavelength for λ.

$k \equiv \frac{2\pi}{\lambda}$ is wavenumber, # waves in (dimensional) length of 2π.

e.g., if $\lambda = \pi$ (meters) then there are 2 waves in 2π meters.
$k = \frac{2\pi}{\pi \text{ m}} = 2 \text{ m}^{-1}$.

Think: long waves --> small k
short waves --> big k

At a fixed point, phase changes by 2π when time changes by $\frac{\lambda}{c}$
So wave period is: $T \equiv \frac{\lambda}{c}$.

$\nu \equiv \frac{1}{T}$ is frequency, # of oscillations per unit time

$\omega \equiv \frac{2\pi}{T} = 2\pi\nu$ is circular (or radian) frequency.

Since $T = \frac{\lambda}{c} \rightarrow \omega = 2\pi \frac{c}{\lambda} = kc$

$$\therefore \boxed{c = \frac{\omega}{k}}$$

equivalent expressions:

$$\sin\frac{2\pi}{\lambda}(x - ct) \quad \text{or} \quad \sin[k(x - ct)] \quad \text{or} \quad \sin(kx - \omega t)$$

Motion of crests/troughts is motion of a geometric pattern. Fluid parcels do not generally move with the wave pattern, e.g. Rossby waves can propagate westward even though parcels move eastward, e.g., tsunami at sea might propagate at 200 m/s, but fluid velocity in this wave might be only ~ 1 m/s.