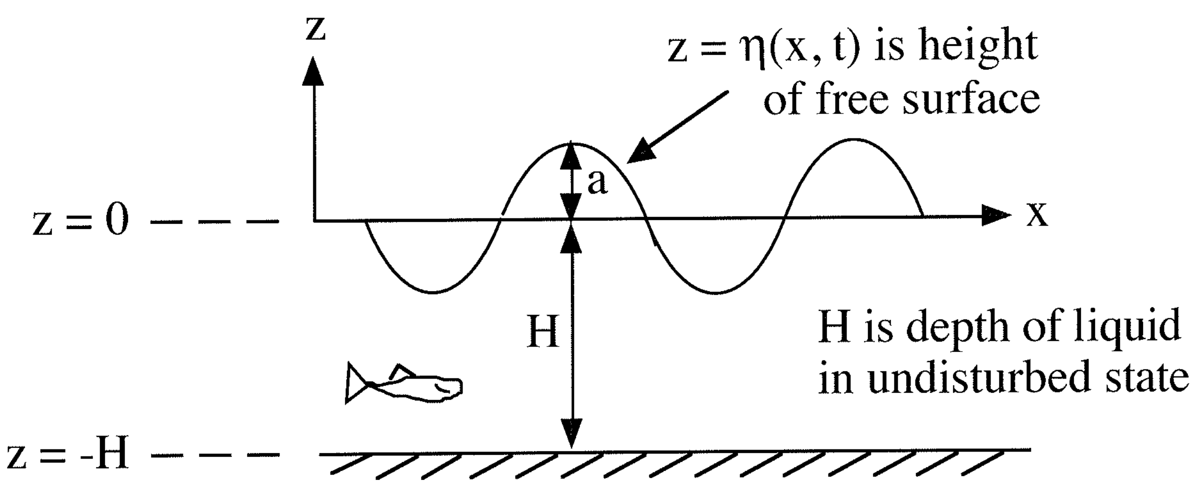
METR 5113, Advanced Atmospheric Dynamics I
Alan Shapiro, Instructor
Monday, 26 November 2018 (lecture 39)

Surface gravity waves

Consider 2-D motion in the x-z plane of a liquid w/ a free surface (e.g. air/sea interface) [a pool of cold air underlying warm air is similar but more complicated]



-- assume disturbance is "infinitesimal", i.e., a is so small that:

$$a \ll H \quad (a/H \ll 1)$$

and $$a \ll \lambda \quad (a/\lambda \ll 1)$$

Because of this, nonlinear accel^n terms are very small compared to local derivs (products of small quantities are really small)

-- neglect friction

-- assume T << rotation period of earth, or equivalently $\omega \gg f$

$\therefore$ can safely neglect Coriolis force.

-- assume fluid was initially at rest (so irrotational) and waves created irrotationally. So from Kelvin's Th^m, the flow will always be irrot. So $\vec{\omega}(t) = 0$

$$\therefore \boxed{\vec{u}(t) = \nabla\phi(t)}$$

-- assume flow is incompressible, $\nabla \cdot \vec{u} = 0$

Substituting in $\vec{u} = \nabla\phi$ we get:

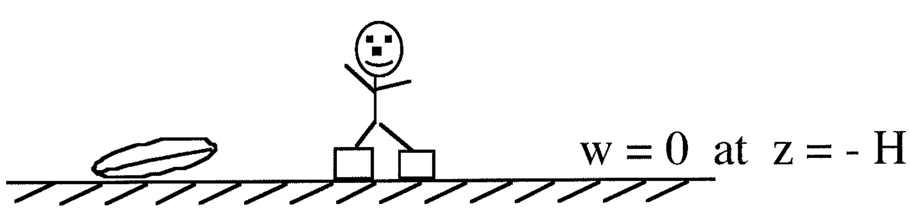
$$\nabla \cdot \nabla\phi = 0$$

$\therefore$ $\nabla^2\phi = 0$ Laplace's eqn.

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0}$$

Need b.c.s to solve it.

At bottom: impermeability cond^n:



$$\therefore \boxed{\frac{\partial\phi}{\partial z} = 0 \text{ at } z = -H}$$

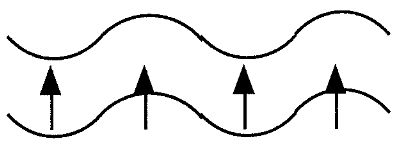
On top (free surface): a kinematic b.c. and a dynamic b.c.

<u>Top Kinematic b.c.</u>: A fluid element on free sfc remains on that sfc no matter how sfc moves or deforms. $\therefore\ z_{\text{parcel on sfc}} = \eta$

$$\therefore\quad w_{\text{parcel on sfc}} = \frac{D}{Dt} z_{\text{parcel on sfc}} = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}$$

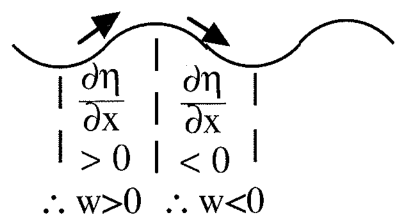
$$\therefore\quad \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \qquad \text{on } z = \eta(x,t)$$

sfc moving upward:



$\partial\eta/\partial t > 0\ \therefore\ w > 0$

Stationary sfc with u>0



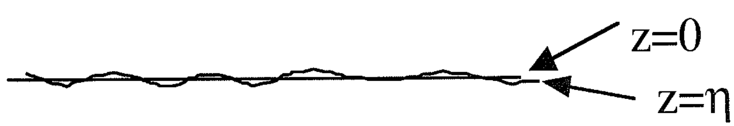
For infinitesimal waves, neglect products of small quantities (nonlinear terms). Neglect $u\,\partial\eta/\partial x$.

$$\therefore\quad \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{on } z = \eta(x,t) \quad [\text{still nonlinear: } \eta \text{ is affected by flow}]$$

Exapnd l.h.s. in a Taylor series about z = 0:

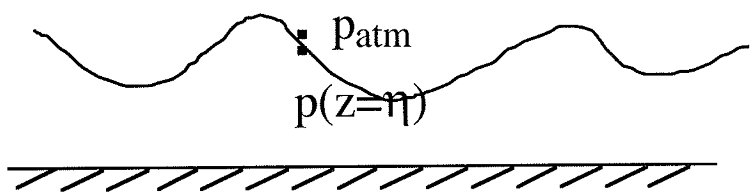
$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \left.\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)\right|_{z=0}\eta + \ldots$$

[neglect these terms since η is small]



$\therefore$ $\boxed{\dfrac{\partial\phi}{\partial z} = \dfrac{\partial\eta}{\partial t} \text{ on } z = 0}$ Linearized kinematic b.c. on free sfc.
[Neglected a nonlinear term and put b.c. at z=0 instead of z=η]

Dynamic b.c. on free surface: pressure is continuous across air/sea interface. So pressure on liquid side of interface = atm pressure.



$$p(z = \eta(x,t)) = p_{atm}$$

[We implicitly used dynamic b.c. in the open barrel problem]

Translate this b.c. into a b.c. on ϕ, using Bernoulli's eq^n for unsteady, irrot flow:

$$\frac{\partial\phi}{\partial t} + \frac{q^2}{2} + \frac{p}{\rho} + gz = C \quad \text{(same const everywhere)}$$

Apply it on free sfc $z=\eta$ (liquid side), w/ $p=p_{atm}$, and neglect q^2:

$$\therefore \quad \frac{\partial\phi}{\partial t} + \frac{p_{atm}}{\rho} + g\eta = C$$

$$\therefore \quad \frac{\partial\phi}{\partial t} + g\eta = \left(C - \frac{p_{atm}}{\rho}\right) = \text{const}$$

Set const = 0. [Flow doesn't care about it. Equivalently, define $\phi_{new} = \phi_{old} -$ const t, and plug into above eq^n. The const cancels out, but since $\vec{u} = \nabla\phi$ and $\nabla\phi_{new} = \nabla\phi_{old}$, the flow is unchanged.]

$$\therefore \boxed{\frac{\partial \phi}{\partial t}} + g\eta = 0 \quad \text{at } z = \eta$$

$\downarrow$

Taylor exp about z = 0:

[neglect since η is small]

$$\therefore \left.\frac{\partial \phi}{\partial t}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial t}\right|_{z=0} + \left.\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)\right|_{z=0} \eta + \text{h.o.t.}$$

$\therefore$ <u>linearized</u> <u>dynamic</u> free sfc b.c. is:

$$\boxed{\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } z = 0}$$

Want to solve $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0$ subject to the above b.c.

Consider a "wavy" pattern for η: $\eta = a\cos(kx - \omega t)$

From either top b.c., suspect $\phi \propto \sin(kx - \omega t)$ w/ a z dependence.

Try: $\phi = f(z)\sin(kx - \omega t)$ [see if it works]

Plug into Laplace's eq^n, get:

$$-k^2 f \sin(kx - \omega t) + \frac{d^2 f}{dz^2}\sin(kx - \omega t) = 0$$

$$\therefore \frac{d^2 f}{dz^2} - k^2 f = 0 \quad \text{a linear const coeff homogeneous ode}$$

Trial soln for f: $f = e^{mz}$. Plug into ode, get:

$$m^2e^{mz} - k^2e^{mz} = 0, \quad \text{divide by } e^{mz}$$

$$m^2 - k^2 = 0$$

$\therefore$ m = k or - k

So general sol^n for f is:

$$f = A\,e^{kz} + B\,e^{-kz}$$

$$\therefore \quad \phi = \left(A\,e^{kz} + B\,e^{-kz}\right)\sin(kx - \omega t)$$