METR 5113, Advanced Atmospheric Dynamics I
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Reminder: please fill out info sheet about exam days/times

A 2nd order tensor S is symmetric if: $S_{ij} = S_{ji}$. It has at most 6 distinct elements.

$$\text{e.g.}\quad S = \begin{pmatrix} 3 & 4 & 5 \\ 4 & 1 & 6 \\ 5 & 6 & 2 \end{pmatrix}$$

A 2nd order tensor A is antisymmetric if: $A_{ij} = -A_{ji}$. It has at most 3 distinct elements since all diagonal elements are 0:

$$A_{11} = -A_{11} \qquad \text{add } A_{11} \text{ to both sides}$$

$$2A_{11} = 0$$

$$\therefore \quad A_{11} = 0$$

Similarly $A_{22} = 0$, $A_{33} = 0$.

$$\text{e.g.}\qquad A = \begin{pmatrix} 0 & 4 & 5 \\ -4 & 0 & -6 \\ -5 & 6 & 0 \end{pmatrix}$$

So, a 2nd order antisymmetric tensor has same number of distinct elements as a vector in 3-D space: 3. Can associate the elements of an antisymmetric tensor w/ comps of a vector, and vice versa.

Let $\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ be a vector.

Define an antisym tensor R associated with $\vec{\omega}$ to be:

$$R_{ij} = -\varepsilon_{ijk}\,\omega_k \qquad \text{or:} \qquad R = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

Any 2nd order tensor B can be written as the sum of a sym tensor and an antisym tensor:

$$\underset{\text{arbitrary}}{B_{ij}} = \underset{\text{sym}}{S_{ij}} + \underset{\text{antisym}}{A_{ij}}$$

Proof:

$$B_{ij} = \tfrac{1}{2} B_{ij} + \tfrac{1}{2} B_{ij}$$

$$= \underbrace{\tfrac{1}{2}\left(B_{ij} + B_{ji}\right)} + \underbrace{\tfrac{1}{2}\left(B_{ij} - B_{ji}\right)}$$

Call whole thing S_{ij} — Need to show it's symmetric

Call whole thing A_{ij} — Need to show it's antisymmetric

$S_{ij} = \frac{1}{2}\left(B_{ij} + B_{ji}\right)$ rename the free indices, get:

$S_{ji} = \frac{1}{2}\left(B_{ji} + B_{ij}\right) = \frac{1}{2}\left(B_{ij} + B_{ji}\right) = S_{ij}$ So, yes! S is sym.

swap order of terms

$A_{ij} \equiv \frac{1}{2}\left(B_{ij} - B_{ji}\right)$ rename the free indices, get:

$$A_{ji} = \frac{1}{2}\left(B_{ji} - B_{ij}\right) = \frac{1}{2}\left(-B_{ij} + B_{ji}\right) = -\frac{1}{2}\left(B_{ij} - B_{ji}\right) = -A_{ij}$$

swap order of terms factor out minus sign

So yes! A is antisym.

<u>Contraction</u>: 2 free indices in a tensor eq^n are set equal to each other (so become a pair of dummy indices). So we're summing!

e.g. Consider 3rd order tensor eq^n $A_{ijk} = u_i \frac{\partial u_k}{\partial x_j}$. Contracting j, k yields $A_{ijj} = u_i \frac{\partial u_j}{\partial x_j}$, a 1st order tensor eq^n. Contracting i, j in A_{ijk} yields $A_{iik} = u_i \frac{\partial u_k}{\partial x_i}$, a different 1st order tensor eq^n.

A contraction of a 2nd order tensor yields a <u>trace</u> (sum of diag elements): $B_{ii} = B_{11} + B_{22} + B_{33}$. A <u>scalar</u> (no free indices)

Consider velocity gradient tensor: $\frac{\partial u_i}{\partial x_j}$. A contraction of this 2nd order tensor is the divergence of the velocity field:

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \vec{u}$$

Contracting 2 indices reduces order of a tensor by 2:
3rd order tensor ------contraction----> 1st order tensor (vector)
2nd order tensor ------contraction----> scalar

A special double-contraction. Consider 4th order tensor arising from product of <u>symmetric</u> 2nd order tensor τ w/ <u>antisym</u> 2nd order tensor B, $\tau_{ij} B_{kl}$. Contract i,k and j,l indices, get the scalar $\tau_{ij} B_{ij}$. Can show it's 0:

$\tau_{ij} B_{ij} = \tau_{ji} B_{ij}$ since τ is sym

$= -\tau_{ji} B_{ji}$ since B is antisym

$= -\tau_{ij} B_{ij}$ renaming dummy indices (i --> j, j --> i)

Add $\tau_{ij} B_{ij}$ to both sides.

$\therefore \quad 2\,\tau_{ij} B_{ij} = 0$

$\therefore \quad \tau_{ij} B_{ij} = 0$

$\therefore$ Doubly-contracted product of sym tensor w/ antisym tensor is 0. [Each comp of τ is multiplied by corresponding comp of B and the products are summed. Analogous to integrating product of an even and an odd function over an even interval -- get 0.]

Recall that for any arbitrary 2nd order tensor E that,

$$\underset{}{E_{ij}} = \underset{\text{sym}}{S_{ij}} + \underset{\text{antisym}}{A_{ij}}$$

Consider doubly contracted product of E w/ sym tensor τ:

$$E_{ij}\tau_{ij} = S_{ij}\tau_{ij} + \boxed{A_{ij}\tau_{ij}} \text{-->0} \quad \text{(sym times antisym = 0)}$$

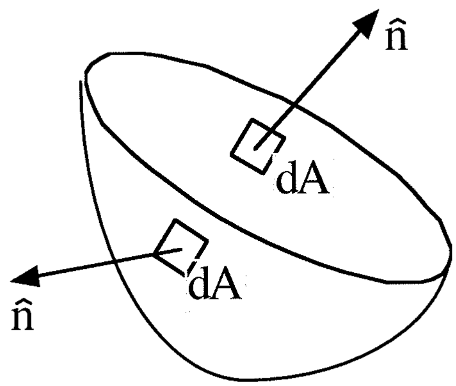
Only symmetric part of E survives in the product:

$$\therefore \quad E_{ij}\tau_{ij} = \frac{1}{2}\left(E_{ij} + E_{ji}\right)\tau_{ij}$$

Integral Theorems

Gauss Theorem.

Consider a volume V bounded by closed surface A w/ local unit outward normal $\hat{n}$. Outward-directed area element is $\hat{n}$ dA. [closed surface -- e.g., a balloon, but NOT a cup]



and consider a tensor Q of any order such that Q and it's derivs are continuous in V. Then Gauss Th^m says:

$$\int_V \frac{\partial Q}{\partial x_i} dV = \int_A Q n_i dA$$

volume integral — area integral (boundary integral)

Here $n_i dA = \hat{e}_i \cdot \hat{n} \, dA$, the projection of $\hat{n}\,dA$ in the $\hat{e}_i$ dir^n.

Gauss Th^m is a 3-D version of Fundamental Th^m of Integral Calculus:

$$\int_a^b \frac{dG}{dx} dx = G(b) - G(a)$$

Suppose Q is a vector w/ components Q_j , then:

$$\int_V \frac{\partial Q_j}{\partial x_i} dV = \int_A Q_j n_i \, dA \qquad \text{[9 eqns]}$$

Contract i, j indices (sum 3 of the 9 eqns), get:

$$\int_V \frac{\partial Q_i}{\partial x_i} dV = \int_A Q_i n_i dA$$ or in vector form:

$$\boxed{\int_V \nabla \cdot \vec{Q} \, dV = \int_A \vec{Q} \cdot \hat{n} \, dA}$$ Divergence Th^m

Net divergence of $\vec{Q}$ summed in a volume = Net outflux of $\vec{Q}$ through the surface bounding that volume