METR 5113, Advanced Atmospheric Dynamics I
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Friday, 30 November 2018 (lecture 41)

- 4 handouts: Info about final exam, Answers to Exam 2, Answers to p.s. 5, and Particle orbits in a sfc gravity wave (from Kundu)

<u>Surface gravity waves (cont^d)</u>

recall that:

$$\omega = \sqrt{gk\tanh(kH)}\,, \qquad c = \sqrt{\frac{g}{k}\tanh(kH)}$$

and that for "shallow water": $kH \ll 1$ so we're led to

$$\omega_{\text{shallow}} = k\sqrt{gH}\,, \qquad c_{\text{shallow}} = \sqrt{gH}$$

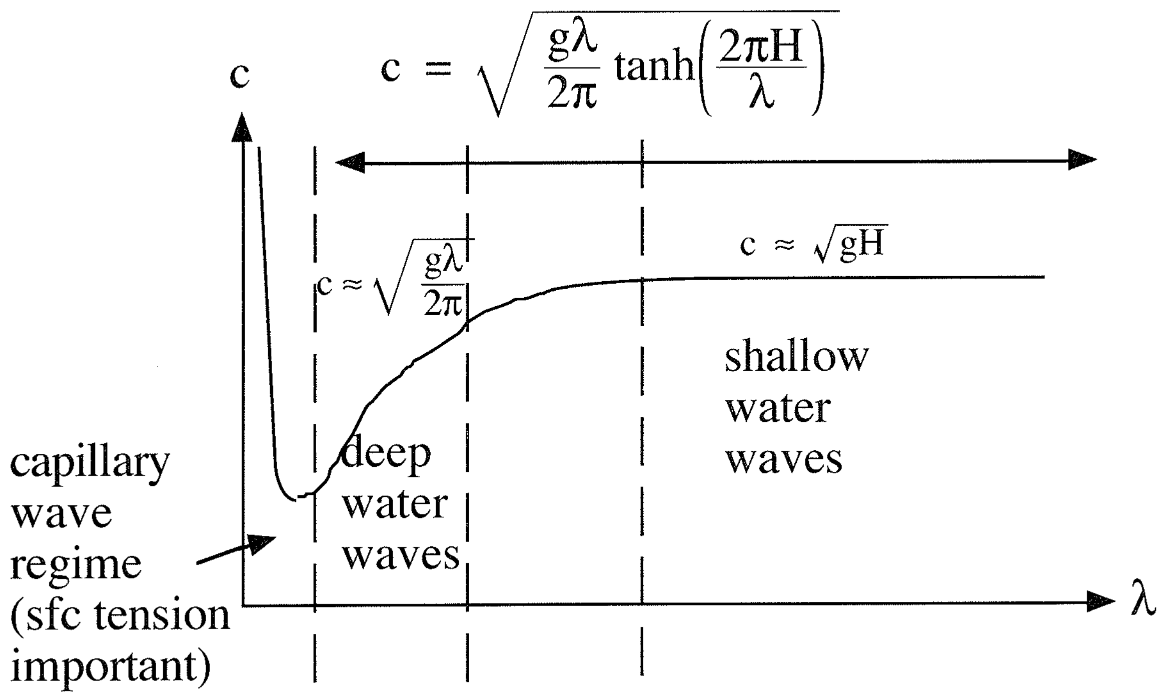
Now consider pressure in shallow water conditions:

$$\frac{p_{\text{shallow}}}{\rho} = \text{const} - gz + \frac{a\underbrace{\omega^2}_{\approx k^2 gH}}{k}\,\frac{\overbrace{\cosh[k(z+H)]}^{\approx 1}}{\underbrace{\sinh(kH)}_{\approx kH}}\cos(kx-\omega t)$$

$$= \text{const} - gz + ag\cos(kx-\omega t)$$

$$= \text{const} - g(z-\eta) \quad \underline{\text{Hydrostatic pressure distribution.}}$$

Now look at phase speed c for the general surface wave case (deep/shallow/whatever):



"deep" or "shallow" depends on λ relative to H. Water that's 100 m deep is "deep" for λ = 10m but "shallow" for λ = 1000 m.

Derive <u>streamfunction</u> [recall this is a 2D incomp flow]

$$\frac{\partial\psi}{\partial z} = u = \frac{\partial\phi}{\partial x} = a\omega \frac{\cosh[k(z+H)]}{\sinh(kH)} \cos(kx - \omega t)$$

integrate w.r.t. z:

$$(1) \quad \psi = \frac{a\omega}{k} \frac{\sinh[k(z+H)]}{\sinh(kH)} \cos(kx - \omega t) + F(x,t)$$

Similarly, $\frac{\partial\psi}{\partial x} = -w = \ldots$ Integrate w.r.t. x, to get:

$$(2) \quad \psi = \frac{a\omega}{k} \frac{\sinh[k(z+H)]}{\sinh(kH)} \cos(kx - \omega t) + G(z,t)$$

F(x,t) = G(z,t) but f^n of x can't be a f^n of z -- so no x or z dependence. So $F(x,t) = G(z,t) = E(t)$. But $E(t)$ is <u>irrelevant</u> since u, w only care about spatial derivs of ψ. So take $E(t) = 0$.

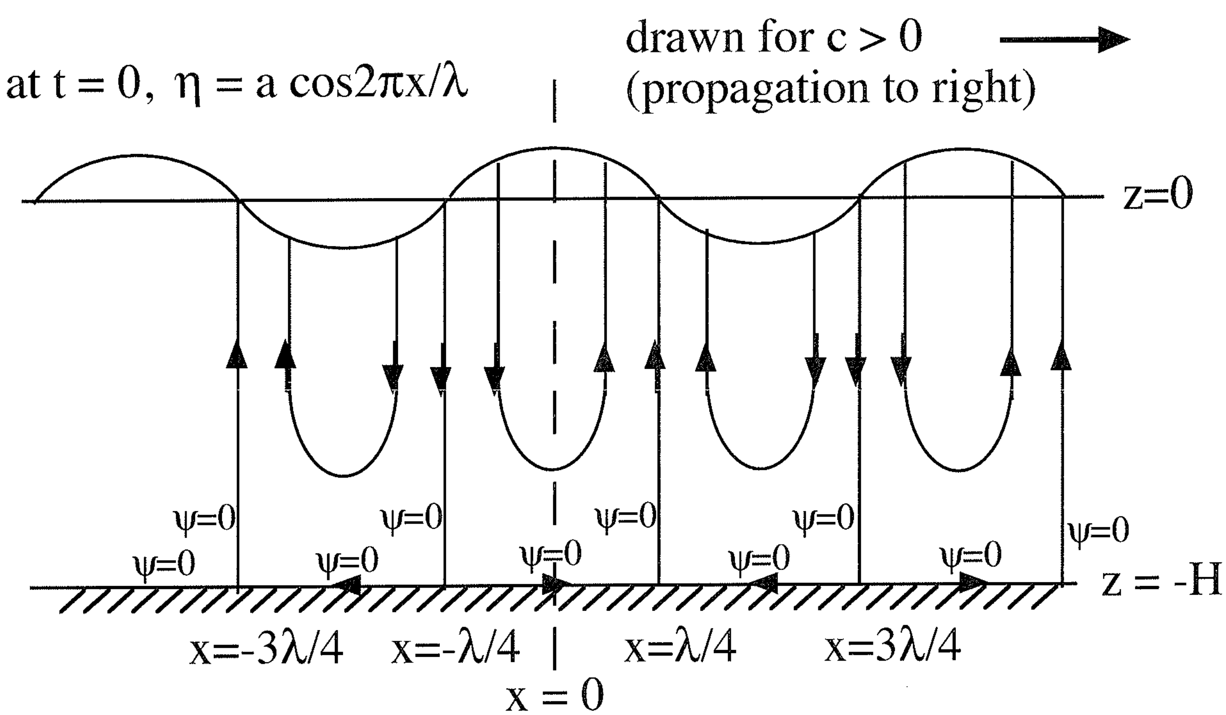
$$\therefore \quad \boxed{\psi = \frac{a\omega}{k}\frac{\sinh[k(z+H)]}{\sinh(kH)}\cos(kx-\omega t)}$$

Graph streamlines (ψ = const) at t = 0.

$\psi = 0$ for: $z = -H$ and for: $\boxed{k}\, x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\downarrow$ $2\pi/\lambda$

$$x = \pm\frac{\lambda}{4}, \pm\frac{3\lambda}{4}, \pm\frac{5\lambda}{4}, \ldots$$



[Get dirn of flow (arrows) from soln for u or w, or consider: for pattern moving toward right, η is rising to right of crest (so $w>0$ there) and η is falling to left of crest (so $w<0$ there). Arrows on bottom give sense of horiz conv/div needed to support this w field. Clearly parcel velocity differs from phase speed.].

Trajectory eqns:

$$\frac{dx}{dt} = u[x(t), z(t), t], \qquad \frac{dz}{dt} = w[x(t), z(t), t] \ .$$

Parcels undergo small oscillations about mean position, x_0, z_0, so can safely approximate the right hand sides of traj eqns as:

$$u[x(t), z(t), t] \approx u(x_0, z_0, t)$$

$$w[x(t), z(t), t] \approx w(x_0, z_0, t)$$

Solution of the approx traj eqns are ellipses (Kundu figs 7.5, 7.6)

Group Velocity

Consider 2 waves of equal amplitude and slightly different frequency and wavelength moving in same direction:

$$\omega_1 = \omega + \Delta\omega, \qquad k_1 = k + \Delta k$$

$$\omega_2 = \omega - \Delta\omega, \qquad k_2 = k - \Delta k$$

assume $\dfrac{\Delta\omega}{\omega} << 1, \qquad \dfrac{\Delta k}{k} << 1$

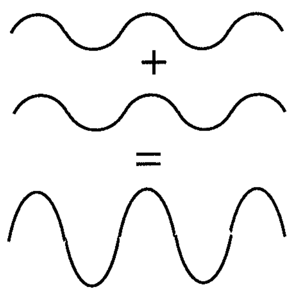
Because of dispersion relation, $\Delta\omega$ is related to Δk.

Mean frequency is: $\dfrac{\omega_1 + \omega_2}{2} = \dfrac{\omega + \Delta\omega + \omega - \Delta\omega}{2} = \omega$

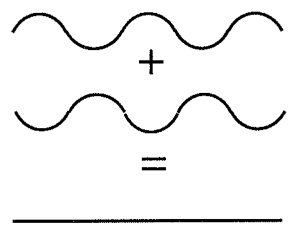
Mean wavenumber is: $\dfrac{k_1 + k_2}{2} = \ldots = k$

Where the waves are in phase (or nearly so) they combine to form a wave of twice amplitude. Where they're out of phase, they kill each other off.

2 waves in phase:



2 waves out of phase:



$$\eta = a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t)$$

$$= a\cos(kx - \omega t + \Delta k x - \Delta\omega t) + a\cos(kx - \omega t - (\Delta k x - \Delta\omega t))$$

$$= a\cos(kx - \omega t)\cos(\Delta k x - \Delta\omega t) \boxed{- a\sin(kx - \omega t)\sin(\Delta k x - \Delta\omega t)}$$

cancellation

$$+ a\cos(kx - \omega t)\cos(\Delta k x - \Delta\omega t) \boxed{+ a\sin(kx - \omega t)\sin(\Delta k x - \Delta\omega t)}$$

$$= 2a\cos(kx - \omega t)\cos(\Delta k x - \Delta\omega t)$$

$$\therefore \ \eta = A \boxed{\cos(kx - \omega t)} \qquad \text{where } A \equiv 2a\cos(\Delta k x - \Delta\omega t)$$

A ↓ effective amplitude; $\cos(kx - \omega t)$ ↓ carrier wave (mean wave)

Effective amplitude A is itself a wave with wavelength

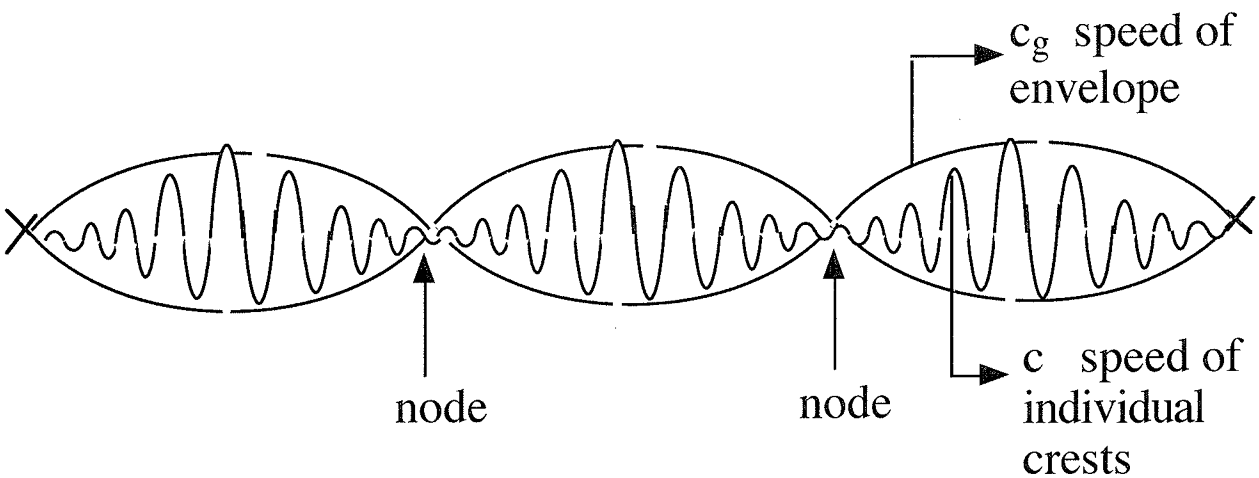
$$\lambda_{\text{amplitude}} = \frac{2\pi}{\Delta k} >> \frac{2\pi}{k} = \lambda_{\text{carrier wave}}$$

A propagates at speed $\frac{\Delta\omega}{\Delta k}$ where $\Delta\omega$ is related to Δk by dispersion rel^n. For small Δk, $\frac{\Delta\omega}{\Delta k} \to \frac{d\omega}{dk}$. Define $c_g \equiv \frac{d\omega}{dk}$

or $\quad \vec{c}_g \equiv \frac{d\omega}{dk}\,\hat{i}$ $\quad$ Group velocity. A vector.

c is phase speed of crests [not a vector, see fig. 7.3 Kundu)

c_g is speed of envelope of crests.



Energy is trapped between nodes ∴ energy propagates at speed of nodes (speed of envelope), i.e. speed c_g, not phase speed c.

Notion of group velocity is applicabale to any type of wave.

For deep-water sfc waves:

$$\omega = \sqrt{gk}$$

$$c = \sqrt{\frac{g}{k}}$$

$$c_g = \frac{d\omega}{dk} = \frac{1}{2\sqrt{k}}\sqrt{g}$$

$\therefore$ $\boxed{c_g = \frac{1}{2}c}$ [since $c > c_g$, individual crests move through envelope, die at nodes]

For <u>shallow water sfc waves</u>:

$$\omega = k\sqrt{gH}$$

$$c = \sqrt{gH}$$

$$c_g = \frac{d\omega}{dk} = \sqrt{gH}$$

$\therefore$ $\boxed{c_g = c}$

End of our work on surface gravity waves. Now look at internal gravity waves.