METR 5113, Advanced Atmospheric Dynamics I
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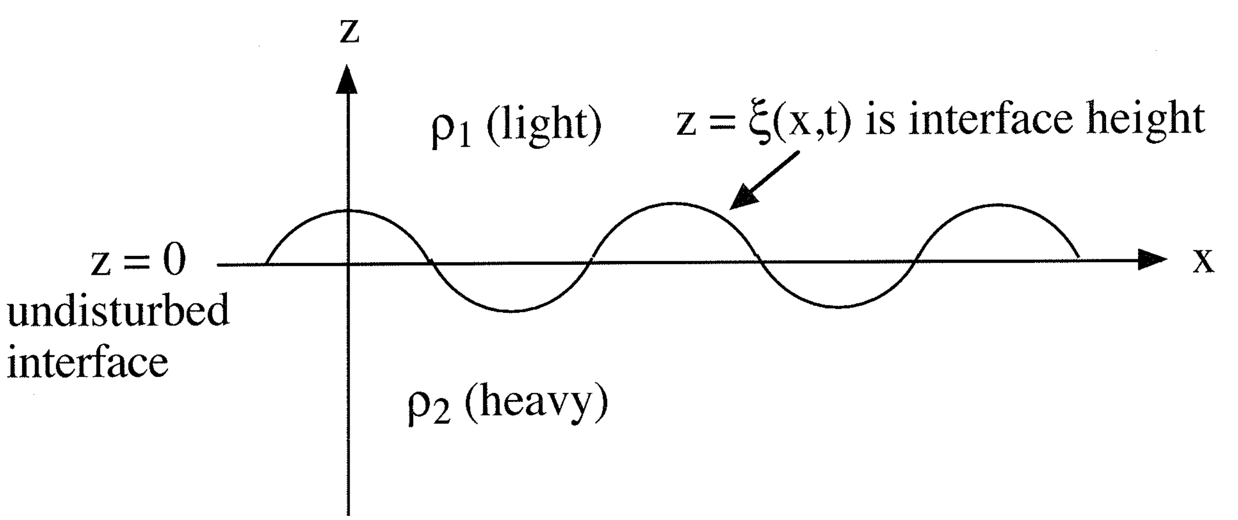
- Please take a few minutes to fill out online course evaluation!

Internal Waves on an Interface

Consider wave motion at interface btw 2 infinitely deep fluids.



Consider $\xi = a \cos(kx - \omega t)$,

or: $\xi = \text{Re}\left[a\, e^{i(kx - \omega t)}\right]$,

or: $\xi = a\, e^{i(kx - \omega t)}$ (with the understanding that we're only considering the real part of it)

Fluid was at rest, then disturbed irrotationally. So Kelvin's thm says motion is <u>always irrot</u>. $\therefore\ \vec{u} = \nabla\phi$ for all t.

$\vec{u} = \nabla\phi_1$ in top fluid, $\vec{u} = \nabla\phi_2$ in bottom fluid

$\nabla \cdot \vec{u} = 0$ in both fluids. $\therefore$ get Laplace's eq^n in both fluids:

$$\nabla^2 \phi_1 = 0 \quad \text{in top fluid}$$

$$\nabla^2 \phi_2 = 0 \quad \text{in bottom fluid}$$

Assume the disturbance dies out far above/below the interface:

$$\phi_1 \to 0 \text{ as } z \to \infty, \qquad \phi_2 \to 0 \text{ as } z \to -\infty.$$

Seek sol^{ns} of form:

$$\phi_1 = \text{Re}\left[b(z)\, e^{i(kx - \omega t)}\right], \qquad \phi_2 = \text{Re}\left[c(z)\, e^{i(kx - \omega t)}\right]$$

where <u>b and c can be complex.</u> Plug into Laplace's eq^n, get:

$\dfrac{d^2b}{dz^2} - k^2 b = 0$	$\dfrac{d^2c}{dz^2} - k^2 c = 0$
$\therefore\ b = A\, e^{kz} + B\, e^{-kz}$	$\therefore\ c = C\, e^{kz} + D\, e^{-kz}$
want $\phi_1 \to 0$ as $z \to \infty$	want $\phi_2 \to 0$ as $z \to -\infty$
$\therefore A = 0$	$\therefore D = 0$
$\therefore\ \phi_1 = B\, e^{-kz}\, e^{i(kx - \omega t)}$	$\therefore\ \phi_2 = C\, e^{kz}\, e^{i(kx - \omega t)}$

Linearized kinematic b.c. on interface:

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \xi}{\partial t} \text{ at } z = 0 \quad \text{and} \quad \frac{\partial \phi_2}{\partial z} = \frac{\partial \xi}{\partial t} \text{ at } z = 0$$

These yield: $B = i\frac{a\omega}{k}$ and $C = -i\frac{a\omega}{k}$

$$\therefore \quad \phi_1 = i\frac{a\omega}{k} e^{-kz} e^{i(kx-\omega t)}$$

$$= i\frac{a\omega}{k} e^{-kz}\left[\cos(kx-\omega t) + i\sin(kx-\omega t)\right]$$

$$= \frac{a\omega}{k} e^{-kz}\left[i\cos(kx-\omega t) - \sin(kx-\omega t)\right]$$

Take real part of it:

$$\boxed{\phi_1 = -\frac{a\omega}{k} e^{-kz}\sin(kx-\omega t)}$$

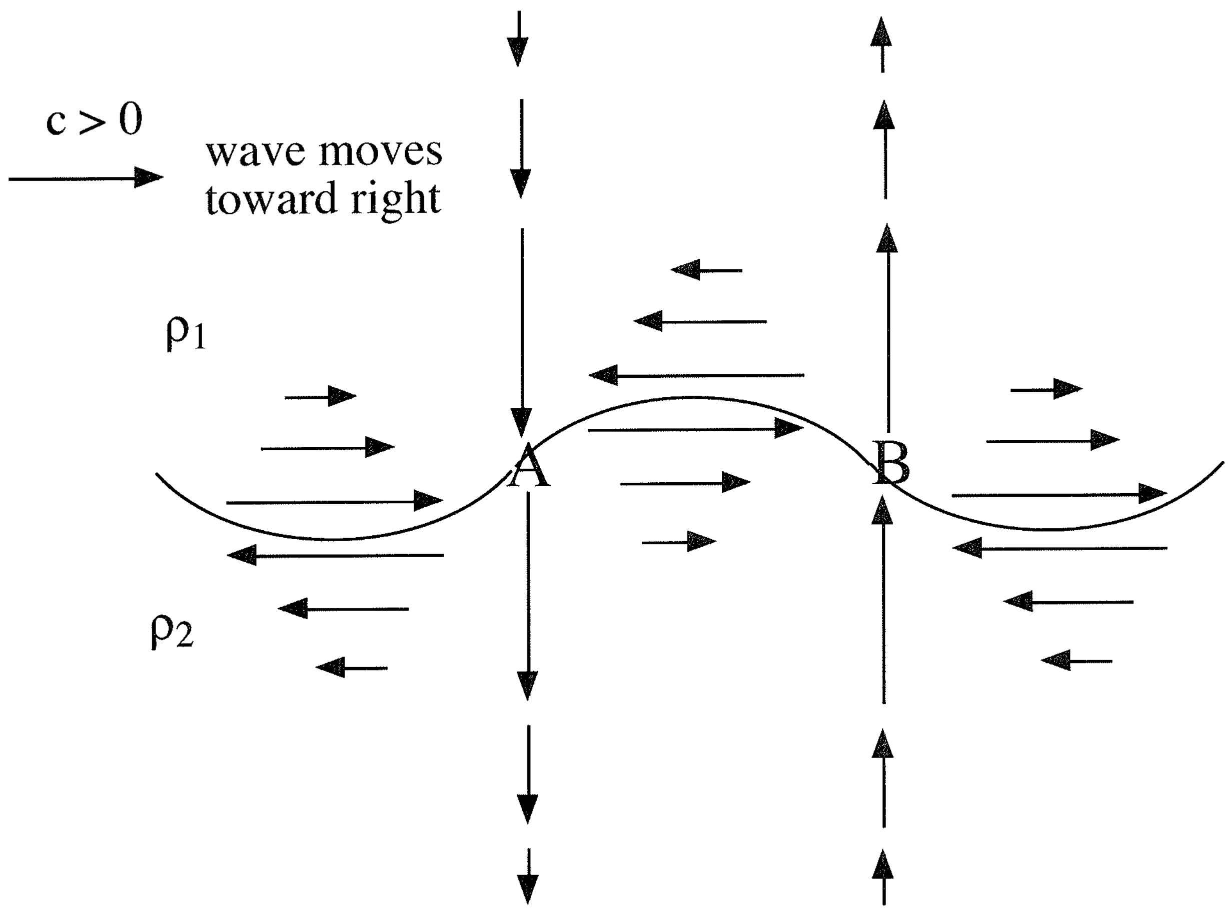
Similarly, find:

$$\boxed{\phi_2 = \frac{a\omega}{k} e^{kz}\sin(kx-\omega t)}$$

So $u_1 = \partial\phi_1/\partial x = -a\omega\, e^{-kz}\cos(kx-\omega t)$

$$u_2 = \partial\phi_2/\partial x = a\omega\, e^{kz}\cos(kx-\omega t)$$

Get w from $w = \partial\phi/\partial z$. But can use u and mass cons^n to deduce behavior of w. Since $\partial w/\partial z = -(\partial u/\partial x + \partial v/\partial y)$, horiz divergence ($\partial u/\partial x + \partial v/\partial y > 0$) is associated with $\partial w/\partial z < 0$. Horiz convergence is associated with $\partial w/\partial z > 0$. Also keep in mind that w goes to 0 far above/below interface. So:



w pattern lets interface propagate toward right. Descent (e.g, at A) allows trough to propagate toward right. Ascent (e.g., at B) allows crest to propagate toward the right.

At interface, u_1 and u_2 are oppositely directed -- a vortex sheet (infinite horiz vorticity). Lap eq^n not valid right at interface [baroclinic vort production going on -- vort not 0 there]. Now lets go after the dispersion relation.

Apply Bernoulli eq^n at <u>interface</u> $z = \xi$. Neglect q^2, get:

on upper side: $$\frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_1} + g\xi = C_1$$

on lower side: $$\frac{\partial \phi_2}{\partial t} + \frac{p_2}{\rho_2} + g\xi = C_2$$

[Important! C_1 and C_2 may not be the same since there's vorticity on the interface! Not same const everywhere...]

$$p_1 = -\rho_1 \frac{\partial \phi_1}{\partial t} - \rho_1 g \xi + \rho_1 C_1 \qquad \text{at } z = \xi$$

$$p_2 = -\rho_2 \frac{\partial \phi_2}{\partial t} - \rho_2 g \xi + \rho_2 C_2 \qquad \text{at } z = \xi$$

Dynamic b.c.: p is continuous across interface, $\boxed{p_1 = p_2}$ at $z = \xi$.

$$\therefore \rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g\xi = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g\xi + \underset{\substack{\downarrow \\ \text{irrelevant -- set to 0}}}{\text{const}} \qquad \text{at } z = \xi.$$

With $\frac{\partial \phi}{\partial t}$ evaluated at z=0 instead of z=ξ (i.e., linearized), get:

$$\rho_1 \frac{a\omega^2}{k} \cos(kx - \omega t) + \rho_1 ga \cos(kx - \omega t) =$$

$$-\rho_2 \frac{a\omega^2}{k} \cos(kx - \omega t) + \rho_2 ga \cos(kx - \omega t)$$

$$\therefore \quad \text{find that:} \quad \omega = \sqrt{gk\left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right)}$$

Note: if $\rho_1 \to 0$ get $\omega = \sqrt{gk}$ which is dispersion rel^n for <u>sfc gravity waves in deep water</u> (i.e., where top fluid is air).

If density dif $\rho_2 - \rho_1$ is small then ω is small $\therefore$ T is large.
In general, waves on an internal interface have a smaller frequency and phase speed than sfc waves.