METR 5113, Advanced Atmospheric Dynamics I
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- **Please fill out your online course evaluation!**

Internal Waves in a Continuously Stratified Fluid

- Suppose mean density $\bar{\rho}(z)$ decreases continuously w/ height.

- Neglect friction, nonlinear terms and Coriolis force.

Work w/ linearized inviscid Boussinesq eq^{ns} of motion:

(1) $$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

[$\rho' \equiv \rho - \rho_0 \;\rightarrow\; \rho = \rho' + \rho_0$
ρ_0 is const (density at a ref level),
$p' \equiv p - \bar{p}(z) \;\rightarrow\; p = p' + \bar{p}(z)$]

(2) $$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

(3) $$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0}$$

incomp cond^n:

(4) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Thermodynamic energy eq^n for a non-diffusive liquid:

$$\frac{D\rho}{Dt} = 0\,.$$

Define pert from mean density:

$$\rho'' \equiv \rho - \bar{\rho}(z) \,. \qquad \text{So } \rho = \rho'' + \bar{\rho}(z)$$

So thermo eqn becomes:

$$\therefore \quad \frac{\partial(\bar{\rho} + \rho'')}{\partial t} + u\frac{\partial(\bar{\rho} + \rho'')}{\partial x} + v\frac{\partial(\bar{\rho} + \rho'')}{\partial y} + w\frac{\partial(\bar{\rho} + \rho'')}{\partial z} = 0$$

$$\therefore \quad \frac{\partial \rho''}{\partial t} + u\frac{\partial \rho''}{\partial x} + v\frac{\partial \rho''}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} + w\frac{\partial \rho''}{\partial z} = 0$$

linearize it, get:

$$\frac{\partial \rho''}{\partial t} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$

Since $\rho'' \equiv \rho - \bar{\rho}(z) = \rho_0 + \rho' - \bar{\rho}(z)$, we see that $\dfrac{\partial \rho''}{\partial t} = \dfrac{\partial \rho'}{\partial t}$.

So linearized thermo energy eq^n becomes:

$$(5) \qquad \frac{\partial \rho'}{\partial t} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$

Eqns (1) - (5) are 5 eq^{ns} in 5 unknowns. Let's get 1 eq^n for just w. Start by eliminating u, v.

Take $\partial/\partial x$ (1) + $\partial/\partial y$ (2):

$$\frac{\partial}{\partial t}\boxed{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial x^2} - \frac{1}{\rho_0}\frac{\partial^2 p'}{\partial y^2}$$

from (4) it's $-\partial w/\partial z$

(6) $$\nabla_H^2 p' = \rho_0 \frac{\partial^2 w}{\partial t \, \partial z}$$

To eliminate ρ', take ∂/∂t (3):

$$\frac{\partial^2 w}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial t\,\partial z} - \frac{g}{\rho_0}\boxed{\frac{\partial \rho'}{\partial t}} \rightarrow -w\, d\bar{\rho}/dz \text{ from (5)}$$

$$\frac{\partial^2 w}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial t\,\partial z} + \frac{g}{\rho_0}\frac{d\bar{\rho}}{dz} w$$

Define Brunt-Väisälä frequency N(z) such that $N^2 \equiv -\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}$

(7) $$\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial t\,\partial z} = -\frac{\partial^2 w}{\partial t^2} - N^2 w$$

Eqns (6) and (7) are two eqns in two unknowns. To eliminate p' from (6) and (7), take ∇_H^2 (7):

$$\frac{1}{\rho_0}\frac{\partial^2}{\partial t\,\partial z}\boxed{\nabla_H^2 p'} = -\frac{\partial^2}{\partial t^2}\nabla_H^2 w - N^2 \nabla_H^2 w$$

↓ $\rho_0 \frac{\partial^2 w}{\partial t\,\partial z}$ from (6)

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2 w}{\partial z^2} = -\frac{\partial^2}{\partial t^2}\nabla_H^2 w - N^2 \nabla_H^2 w$$

use $\nabla^2 = \nabla_H^2 + \frac{\partial^2}{\partial z^2}$

$$\boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2(z)\nabla_H^2 w = 0}$$

<u>Internal wave eq^{n}.</u> 4th order linear homogeneous PDE.

Examine simplest case, where <u>N(z) = const.</u> [If N is not constant but waves have wavelength that is much smaller than vertical scale over which N changes appreciably then waves behave as if N was constant. In this case we can approximate N as constant and refer to the waves as "small scale internal gravity waves"]

So, with N = const the pde has const coefficients. Try a plane wave solution:

$$w = w_0 e^{i(kx + ly + mz - \omega t)} = w_0 e^{i(\vec{K}\cdot\vec{x} - \omega t)}$$

$$\vec{K} \equiv k\hat{i} + l\hat{j} + m\hat{k} \quad \text{is wavenumber vector.}$$

$$\vec{k}_H \equiv k\hat{i} + l\hat{j} \quad \text{is horizontal wavenumber vector.}$$

$$\vec{K} = \vec{k}_H + m\hat{k}$$

$$\left|\vec{K}\right| = \sqrt{k^2 + l^2 + m^2}\,.$$

$$k_H \equiv \left|\vec{k}_H\right| = \sqrt{k^2 + l^2}\,.$$

Plug expression for w into internal wave eq^{n}, get:

$$(-i\omega)^2\left[(ik)^2 + (il)^2 + (im)^2\right] + N^2\left[(ik)^2 + (il)^2\right] = 0$$

$$\therefore \quad -\omega^2(-k^2 - l^2 - m^2) - N^2(k^2 + l^2) = 0$$

$$\therefore \quad \left|\vec{K}\right|^2 \omega^2 - N^2 k_H^2 = 0$$

$$\omega = \sqrt{N^2 \frac{k_H^2}{\left|\vec{K}\right|^2}}$$

If $\partial\bar{\rho}/\partial z < 0$ (statically stable case) then $N^2 > 0$ and $\omega = N\dfrac{k_H}{\left|\vec{K}\right|}$

is a real number. So $e^{i(\vec{K}\cdot\vec{x}-\omega t)}$ is a propagating wave. If $\partial\bar{\rho}/\partial z > 0$ (statically unstable case), ω would be imaginary and $e^{i(\vec{K}\cdot\vec{x}-\omega t)}$ could blow up exponentially with t.