METR 5113, Advanced Atmospheric Dynamics I
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2 handouts: Prob set 1 answers. Streamlines vs trajectories

Kinematics (continued)

Can express DF/Dt (a Lagrangian concept) in Eulerian notation. At time t a blob is at $\vec{x}$ with F value of $F(\vec{x}, t)$. A short time Δt later, same blob is at $\vec{x} + \overrightarrow{\Delta x}$ with an F value $F(\vec{x} + \overrightarrow{\Delta x}, t + \Delta t)$. The $\overrightarrow{\Delta x}$ and Δt are related by: $\overrightarrow{\Delta x} = \vec{u}\Delta t$ (in limit $\Delta t \to 0$), or $\Delta x_i = u_i \Delta t$. The change in F of the blob over time interval Δt is:

$$\Delta F = F(\vec{x} + \overrightarrow{\Delta x}, t + \Delta t) - F(\vec{x}, t)$$

So we can write the total derivative of F as,

Taylor expand this term

$$\frac{DF}{Dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta F}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(\vec{x} + \overrightarrow{\Delta x}, t + \Delta t) - F(\vec{x}, t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{F(\vec{x},t) + \frac{\partial F}{\partial x_i}\boxed{\Delta x_i} + \frac{\partial F}{\partial t}\Delta t + \text{higher order terms} - F(\vec{x},t)}{\Delta t}$$

$$\therefore \quad \boxed{\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u_i \frac{\partial F}{\partial x_i}}$$

or: $$\boxed{\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \vec{u} \cdot \nabla F} \qquad (*)$$

total deriv — local deriv — advection term

L.h.s. is a Lagrangian expression, r.h.s. is an Eulerian expression.

Note: if a measurement device (e.g., thermometer) is on a moving platform (e.g. car or plane) then the time rate of change of F as measured by that device can be expressed as:

$$\left.\frac{DF}{Dt}\right|_{device} = \frac{\partial F}{\partial t} + \vec{u}_{device} \cdot \nabla F$$

Setting $F = u_j$ in (*) yields an expression for the acceleration:

$$\frac{Du_j}{Dt} = \frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} \qquad [j = 1, 2 \text{ or } 3]$$

or,

$$\frac{Du_j}{Dt} = \frac{\partial u_j}{\partial t} + (\vec{u} \cdot \nabla) u_j$$

or in vector form:

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u}$$

Acceleration is <u>linear</u> in Lag^n description but <u>nonlinear</u> in Eulerian description. This inertial nonlinearity makes analytic solutions of eq^{ns} of motion in Eulerian description difficult (p.g.f. and diffusion make analytic sol^{ns} difficult in Lag^n description). NWP models are usually Eulerian, but some are hybrid: Lag^n description for acceleration and Eulerian for other terms.

Note: if we know $\vec{u}$ as a f^n of grid coordinates x, y, z and t (i.e., Eulerian data), can still solve for trajectory. Trajectory eq^{ns} are:

$$\frac{D\vec{x}}{Dt} = \vec{u}[\vec{x}(\vec{x}_0, t), t]$$

It's a hybrid Lagrangian/Eulerian notation. $\vec{u}$ is expressed as a f^n of the fixed points that happen to coincide with location $\vec{x}$ of the parcel. At each time t work with a different set of fixed points.

Can also write: $\frac{D\vec{x}}{Dt} = \vec{u}[\vec{x}(t), t]$ where $\vec{x}_0$ is implied.

or in components:

$$\frac{Dx}{Dt} = u\left[x(t), y(t), z(t), t\right],$$

$$\frac{Dy}{Dt} = v\left[x(t), y(t), z(t), t\right],$$

$$\frac{Dz}{Dt} = w\left[x(t), y(t), z(t), t\right].$$

3 coupled ordinary differential equations (odes). The 3 $const^s$ of integration are fixed by choice of $\vec{x}_0$. Can solve odes <u>numerically</u> (e.g., Runge-Kutta) or in simple cases solve them analytically.

If $\frac{\partial}{\partial t}$ (dep variables) = 0 then we have a <u>steady state</u>.

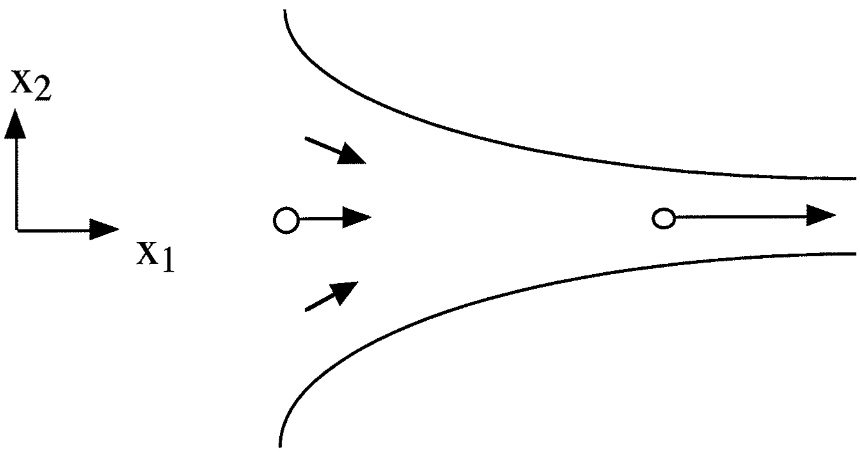
Note: air parcels in some steady state flows can still accelerate:

$$\frac{Du_j}{Dt} = \boxed{\frac{\partial u_j}{\partial t}} + u_i \frac{\partial u_j}{\partial x_i}$$

0 in steady state

.

e.g., steady flow through a canyon [vertical cliffs]:
Suppose we know that the following flow is in a steady state:

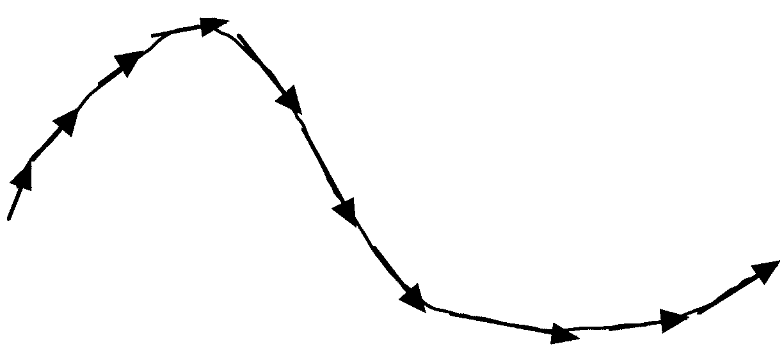


Show that parcels accelerate along centerline:

$$\frac{Du_1}{Dt} = \underbrace{\frac{\partial u_1}{\partial t}}_{0\ (s.s)} + u_1\frac{\partial u_1}{\partial x_1} + \underbrace{\boxed{u_2\frac{\partial u_1}{\partial x_2}}}_{0\ (why?)} + \underbrace{u_3\frac{\partial u_1}{\partial x_3}}_{0\ (no\ vert\ motion)}$$

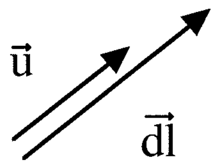
$$\therefore \quad \frac{Du_1}{Dt} = \underset{+}{u_1}\underset{+}{\frac{\partial u_1}{\partial x_1}} > 0$$

<u>Streamline</u>: a line that is everywhere <u>tangent</u> to local velocity vectors (at a given time). It's an Eulerian concept.



A streamline is constructed at a fixed time (snapshot). In general, it does not give history of parcel motion but <u>streamlines and trajectories do coincide if flow is in a steady state.</u>

To derive differential eq^{ns} for streamlines, consider a tiny chunk $\vec{dl}$ of a streamline:



$$\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\,.$$

Since $\vec{dl}$ is $\parallel$ to $\vec{u}$, $\vec{dl} \times \vec{u} = 0$

$$\therefore \quad \hat{i}\left(w\,dy - v\,dz\right) + \hat{j}\left(u\,dz - w\,dx\right) + \hat{k}\left(v\,dx - u\,dy\right) = 0$$

Since each term ( ) must be 0 [dot eq^n with $\hat{i}$, $\hat{j}$, $\hat{k}$ in turn], get:

$$w\,dy - v\,dz = 0 \quad \rightarrow \quad \frac{dz}{w} = \frac{dy}{v}$$

$$u\,dz - w\,dx = 0 \quad \rightarrow \quad \frac{dz}{w} = \frac{dx}{u}$$

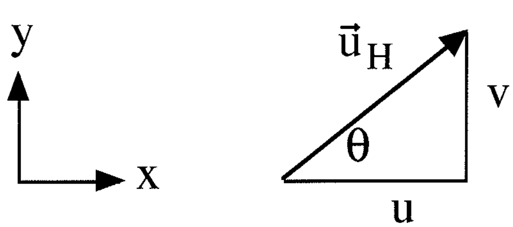
$$v\,dx - u\,dy = 0 \quad \rightarrow \quad \frac{dx}{u} = \frac{dy}{v}\,.$$

So $\boxed{\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}}$ differential eq^{ns} of streamline.

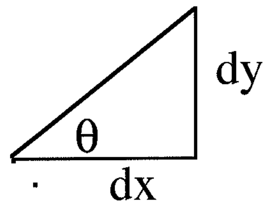
Could also write these as: $\dfrac{dy}{dx} = \dfrac{v}{u}$, $\quad \dfrac{dz}{dx} = \dfrac{w}{u}\,.$

---------- alternate derivation using geometrical considerations:

Consider horiz part of $\vec{u}$ at a point in xy plane, $\vec{u}_H \equiv u\,\hat{i} + v\,\hat{j}$:



streamline through same point looks like:



has <u>same θ</u>! so same tanθ

$$\therefore \quad \frac{dy}{dx} = \frac{v}{u}$$

Similarly, consider xz plane to get $\frac{dz}{dx} = \frac{w}{u}$.

Can rewrite these odes in parametric form.

$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = ds$ where s is a parameter (pseudo-arc length)

÷ by ds:

$$\boxed{\frac{dx}{ds} = u, \quad \frac{dy}{ds} = v, \quad \frac{dz}{ds} = w}$$ Parametric eq^{ns} for streamline

or:

$$\frac{dx}{ds} = u(x, y, z, t^*), \qquad \frac{dy}{ds} = v(x, y, z, t^*), \qquad \frac{dz}{ds} = w(x, y, z, t^*),$$

where t^* is an analysis time, a constant.

3 coupled 1st order odes. The 3 consts of integration allow you to specify a point x_o, y_o, z_o through which the streamline passes.