METR 5113, Advanced Atmospheric Dynamics I
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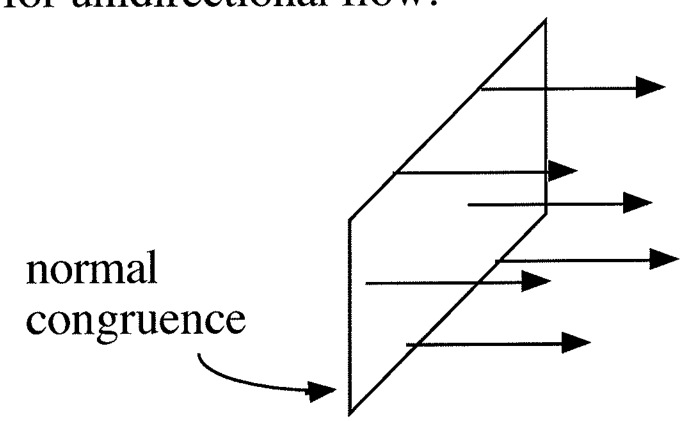
Wednesday, 5 September 2018 (lecture 7)

- Day/time for exams and make-ups: Fridays 9-11 am
- 2 handouts: normal congruence, answers to prob set 1.

Normal congruence

A sfc that is everywhere ⊥ to the flow is a "normal congruence"

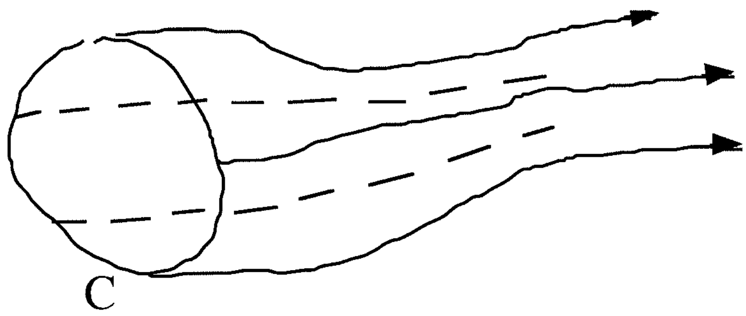
e.g. for unidirectional flow:



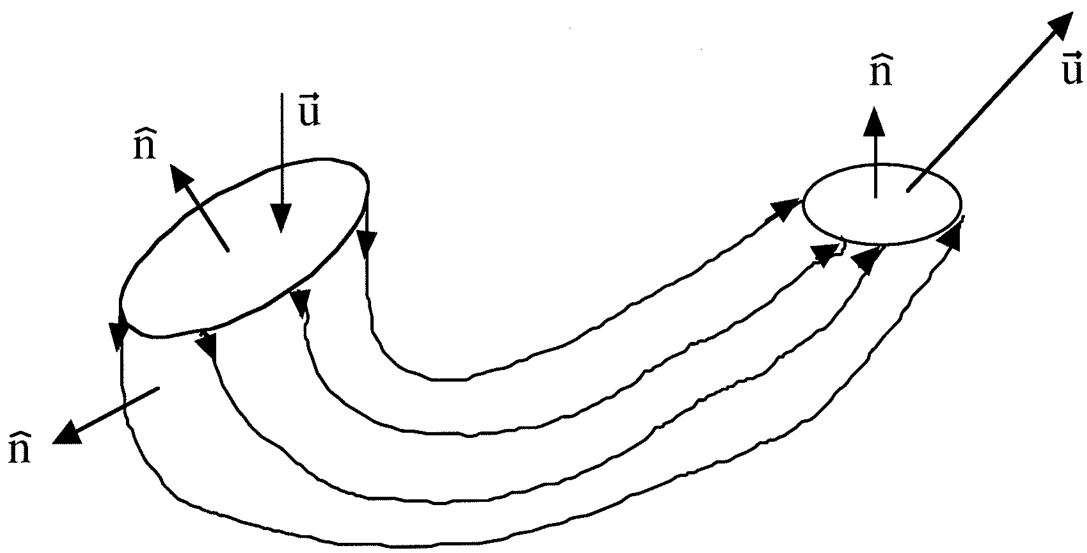
Can such a sfc be constructed in all flows? No! Only in flows that have no helicity, i.e., only if $\vec{u} \cdot (\nabla \times \vec{u}) = 0$. ($\vec{u} \cdot \vec{\omega} = 0$, where $\vec{\omega} \equiv \nabla \times \vec{u}$ is vorticity). See optional handout for details.

Streamtubes

Consider arbitrary closed curve C drawn in the flow at a given time. Through each point of C draw a streamline. The sfc swept out by these streamlines is a streamtube. It's a sfc that's everywhere tangent to local velocity vectors. A streamtube behaves kinematically like a pipe (no flow ⊥ to sidewalls).



Consider volume V enclosed by chunk of a streamtube:



$\hat{n}$ is local unit outward normal to the closed sfc A bounding V (A is streamtube + 2 end-faces). $\hat{n}$ on streamtube is $\perp$ to flow.

If flow is incompressible (3D non-divergent, $\nabla \cdot \vec{u} = 0$) the lhs of div thm $\int_V \nabla \cdot \vec{u}\, dV = \int_A \vec{u} \cdot \hat{n}\, dA$ is 0, so rhs is 0: $\int_A \vec{u} \cdot \hat{n}\, dA = 0$

$$\therefore \quad \int_{face1} \vec{u} \cdot \hat{n}\, dA + \int_{face2} \vec{u} \cdot \hat{n}\, dA + \boxed{\int_{streamtube} \vec{u} \cdot \hat{n}\, dA} = 0$$

0 since $\vec{u} \cdot \hat{n} = 0$ on streamtube

$$\therefore \quad \int_{\text{face1}} \vec{u} \cdot \hat{n}\, dA = -\int_{\text{face2}} \vec{u} \cdot \hat{n}\, dA$$

$\therefore \quad \int \vec{u} \cdot \hat{n}\, dA$ is <u>equal and opposite</u> on the 2 end-faces.

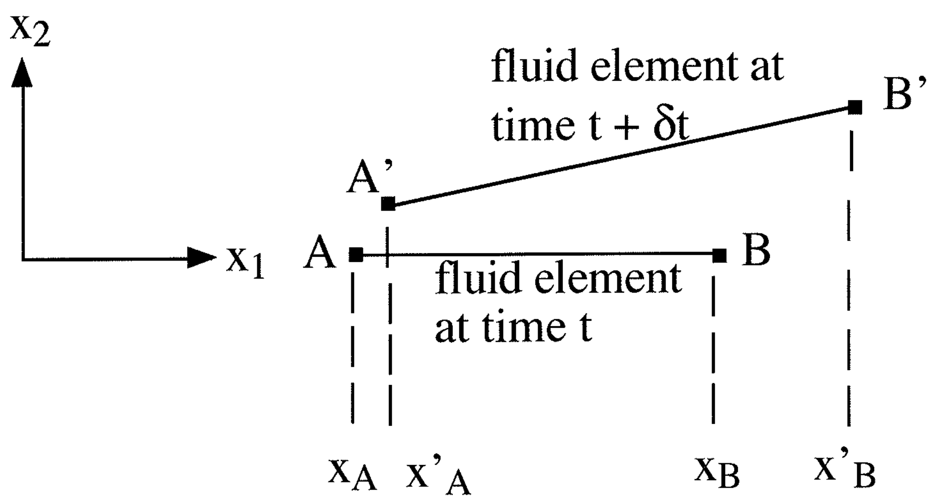
Rate of mass transport <u>inward</u> across one face equals rate of mass transport <u>outward</u> across other face. If a face is small, the velocity is (relatively) large.

<u>Strain rates in a fluid</u>

"Normal" or "linear" rate of strain is <u>rate of change in length of a linear fluid element per unit length of the element</u>. (elements are infinitesimal).

Consider infinitesimal linear fluid element initially aligned in x_1 dir^n. Linear rate of strain of this element is $\lim_{\delta x_1 \to 0} \frac{1}{\delta x_1} \frac{D\delta x_1}{Dt}$.

Relate it to flow field.



[Figure depicts translation, stretching and rotation. Fluid element is so small that velocity varies at most linearly across it, i.e., h.o.t. in Taylor expansion drop out.]

For small δt, elongation of this x_1-oriented element is due only to stretching in x_1 direction. Rotation and translation don't elongate the element. So don't worry about them.

Length of element at time t: $\delta x_1(t) = x_B - x_A$

Length of element at time $t + \delta t$: $\delta x_1(t + \delta t) = x_B' - x_A'$

$$x_B' = x_B + u_B\,\delta t + \text{h.o.t.}$$

$$x_A' = x_A + u_A\,\delta t + \text{h.o.t.}$$

$$\therefore \quad x_B' - x_A' = x_B - x_A + \left(u_B - u_A\right)\delta t + \text{h.o.t.}$$

$$\therefore \quad \delta x_1(t + \delta t) = \delta x_1(t) + \left(u_B - u_A\right)\delta t + \text{h.o.t.}$$

$$\delta x_1(t + \delta t) - \delta x_1(t) = \left(\boxed{u_B} - u_A\right)\delta t + \text{h.o.t.}$$

↓ Taylor expansion for u_B:

$$u_B = u_A + \frac{\partial u_1}{\partial x_1}\delta x_1 + \text{h.o.t.}$$

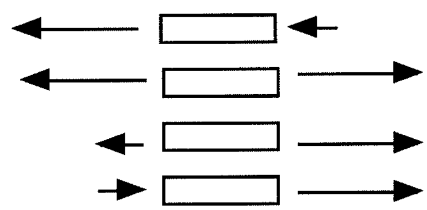
$$\therefore \quad \delta x_1(t + \delta t) - \delta x_1(t) = \frac{\partial u_1}{\partial x_1}\delta x_1\,\delta t + \text{h.o.t.}$$

$\div$ by $\delta x_1\,\delta t$ and let $\delta x_1 \to 0, \delta t \to 0$ to get linear rate of strain:

$$\lim_{\substack{\delta x_1 \to 0 \\ \delta t \to 0}} \frac{\delta x_1(t+\delta t) - \delta x_1(t)}{\delta x_1\, \delta t} = \frac{\partial u_1}{\partial x_1}$$

or, $$\boxed{\frac{1}{\delta x_1}\frac{D\delta x_1}{Dt} = \frac{\partial u_1}{\partial x_1}}$$ (in limit of vanishing δx_1)

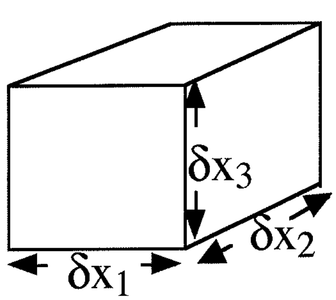
e.g., flows with $\partial u_1/\partial x_1 > 0$:



Similar analysis of x_2- and x_3-oriented blobs leads to:

$$\boxed{\frac{1}{\delta x_2}\frac{D\delta x_2}{Dt} = \frac{\partial u_2}{\partial x_2}},$$ and $$\boxed{\frac{1}{\delta x_3}\frac{D\delta x_3}{Dt} = \frac{\partial u_3}{\partial x_3}}.$$

Volumetric rate of strain: relative rate of change in volume of an infinitesimal fluid box: $\frac{1}{\delta V}\frac{D\delta V}{Dt}$



$$\delta V = \delta x_1\, \delta x_2\, \delta x_3$$

Relate vol rate of strain to the flow field:

$$\frac{1}{\delta V}\frac{D\delta V}{Dt} = \frac{1}{\delta x_1 \delta x_2 \delta x_3}\frac{D}{Dt}\left(\delta x_1 \delta x_2 \delta x_3\right)$$

$$= \frac{1}{\delta x_1 \delta x_2 \delta x_3}\left[\delta x_2 \delta x_3 \frac{D\delta x_1}{Dt} + \delta x_1 \delta x_3 \frac{D\delta x_2}{Dt} + \delta x_1 \delta x_2 \frac{D\delta x_3}{Dt}\right]$$

$$= \frac{1}{\delta x_1}\frac{D\delta x_1}{Dt} + \frac{1}{\delta x_2}\frac{D\delta x_2}{Dt} + \frac{1}{\delta x_3}\frac{D\delta x_3}{Dt}$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} = \nabla \cdot \vec{u}$$

So $\boxed{\frac{1}{\delta V}\frac{D\delta V}{Dt} = \nabla \cdot \vec{u}}$ (in limit of vanishing box size)

So if flow is non-divergent ($\nabla \cdot \vec{u} = 0$), fluid elements do not change their volume (though they can change their shape).