METR 5113, Advanced Atmospheric Dynamics I
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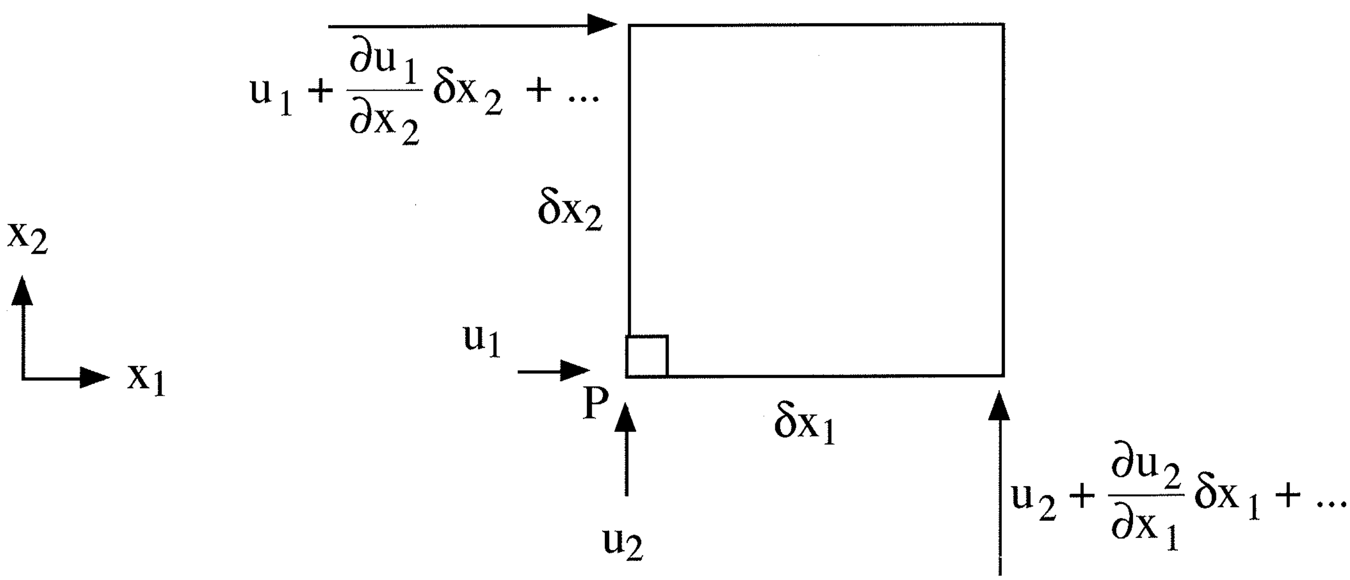
Friday, 7 September 2018 (lecture 8)

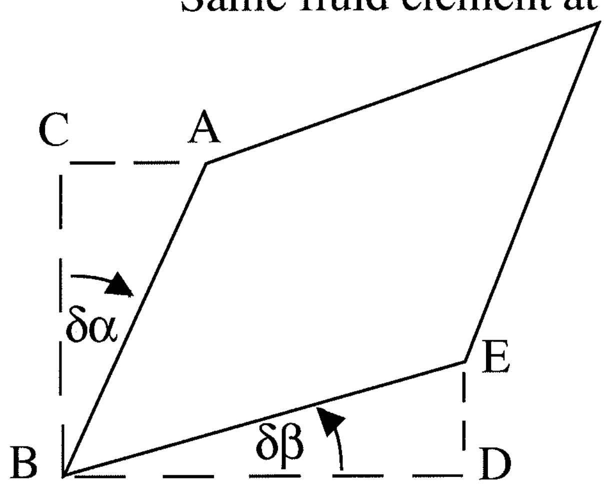
1 handout: Problem set 2

<u>Shear strain rate</u>: Rate of decrease of angle btw 2 mutually perpendicular lines in a fluid element.

Fluid element at time t:



Same fluid element at time t + δt:



So rate of shear strain is: $\lim_{\delta t \to 0} \dfrac{\delta\alpha + \delta\beta}{\delta t}$

Lets relate rate of shear strain to local flow conditions.

from geometry: $\tan\delta\alpha = \dfrac{x_A - x_C}{\delta x_2}$ [for small $\delta\alpha$: $\tan\delta\alpha \approx \delta\alpha$]

$$x_A = x_P + \left(u_1 + \frac{\partial u_1}{\partial x_2}\delta x_2 + \ldots\right)\delta t + \text{h.o.t.}$$

$$x_C = x_B = x_P + u_1 \delta t + \text{h.o.t.}$$

$$\therefore \quad \delta\alpha = \frac{\frac{\partial u_1}{\partial x_2}\delta x_2 \delta t + \text{h.o.t.}}{\delta x_2} = \frac{\partial u_1}{\partial x_2}\delta t + \text{h.o.t.}$$

Similarly can show that $\delta\beta = \dfrac{\partial u_2}{\partial x_1}\delta t + \text{h.o.t.}$

$$\text{So} \quad \boxed{\lim_{\delta t \to 0} \frac{\delta\alpha + \delta\beta}{\delta t} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}}$$

Define <u>rate of strain tensor</u> e by:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Note that: $e_{ij} = e_{ji}$ so e is a symmetric tensor.

Diagonal elements of e:

$$e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$$ = rate of normal strain in x_1 dir^n.

Similarly, $e_{22} = \frac{\partial u_2}{\partial x_2}$, $e_{33} = \frac{\partial u_3}{\partial x_3}$.

∴ Diagonal elements of e are normal strain rates.

∴ Vol strain rate = sum of diag elements of e (trace of e)

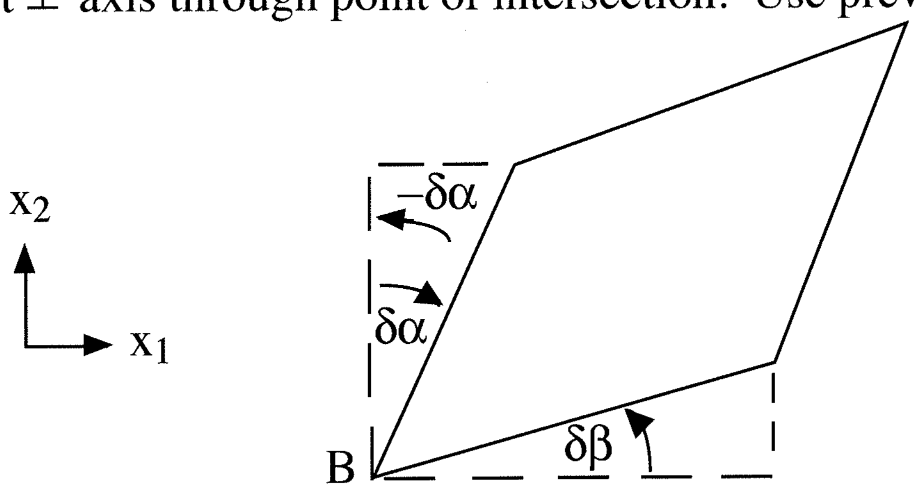
Off-diagonal elements of e:

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}$$ rate of shear strain in x_1x_2 plane .

∴ Off-diagonal elements of e are half the rate of shear strains.

Vorticity

Calculate average angular velocity of 2 ⊥ lines in a fluid element about ⊥ axis through point of intersection. Use prev diagram:



"Average" or "local" angular velocity about x_3 axis through B is:

$$\Omega_3 = \lim_{\delta t \to 0} \frac{1}{2}\left(\frac{\delta\beta + (-\delta\alpha)}{\delta t}\right) = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

Define x_3 comp of <u>vorticity</u> to be <u>twice</u> this local ang velocity:

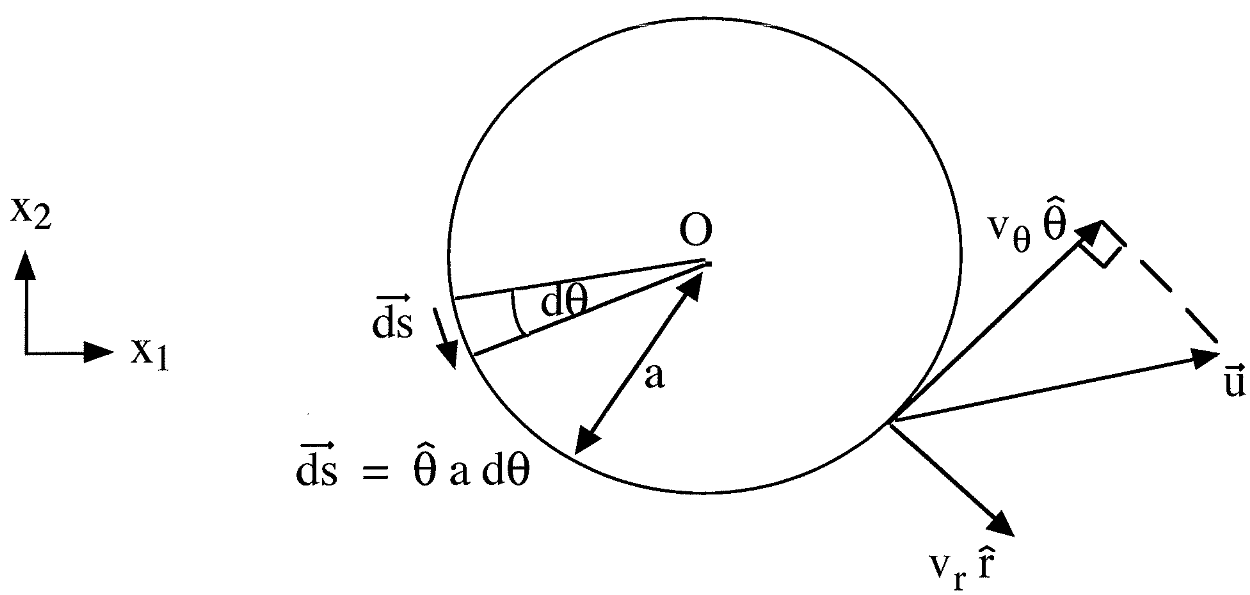
$$\omega_3 \equiv 2\Omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

Get similar formulas for local ang velocity about x_1 and x_2 axes.

$$\vec{\omega} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\hat{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\hat{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\hat{e}_3$$

$$= \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}\hat{e}_i = \nabla \times \vec{u}$$

<u>Another look at vorticity.</u> Draw circle of small (vanishing) radius a around point of interest. Compute <u>ave ang velocity on circle</u>.



$\hat{\theta}$ is unit vector in tangential dir^n.

tangential velocity: $v_\theta = \vec{u} \cdot \hat{\theta}$

angular velocity: $\Omega_3 = \dfrac{v_\theta}{a}$

Calculate the average angular velocity around the circle:

$$\overline{\Omega}_3 = \frac{1}{2\pi}\int_0^{2\pi} \Omega_3 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{v_\theta}{a} d\theta = \frac{1}{2\pi a}\int_0^{2\pi} \vec{u} \cdot \underbrace{\boxed{\hat{\theta}\, d\theta}}_{= \vec{ds}/a}$$

$$= \frac{1}{2\underbrace{\boxed{\pi a^2}}_{=A}} \oint \vec{u} \cdot \vec{ds} = \frac{1}{2A} \oint \vec{u} \cdot \vec{ds}$$ Now apply Stokes Th^m

$$= \frac{1}{2A}\int (\nabla \times \vec{u}) \cdot \hat{n} dA, \qquad \text{where } \hat{n} = \hat{e}_3$$

For teeny-weeny circle (let $a \to 0$) the ave ang velocity is:

$$\overline{\Omega}_3 = \lim_{a \to 0} \frac{1}{2}\frac{1}{A} \int (\nabla \times \vec{u}) \cdot \hat{e}_3 \, dA = \lim_{a \to 0} \frac{1}{2}\frac{1}{A} (\nabla \times \vec{u}) \cdot \hat{e}_3 \, A$$

$$= \frac{(\nabla \times \vec{u}) \cdot \hat{e}_3}{2}$$

$$\therefore \quad \boxed{(\nabla \times \vec{u}) \cdot \hat{e}_3 = 2\overline{\Omega}_3}$$

define vertical vorticity: $\omega_3 \equiv (\nabla \times \vec{u}) \cdot \hat{e}_3$

So <u>vorticity is twice local angular velocity.</u>

It's important to understand the relevant length scales when discussing values of vorticity, e.g. tornado-like value of vorticity $1\ s^{-1}$ can be easily generated by slamming a door. The $1\ s^{-1}$ value is "large" if that value is maintained over an area that's say a few 100 m in radius.