METR 5113, Advanced Atmospheric Dynamics I
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- 1 handout: review of linear algebra

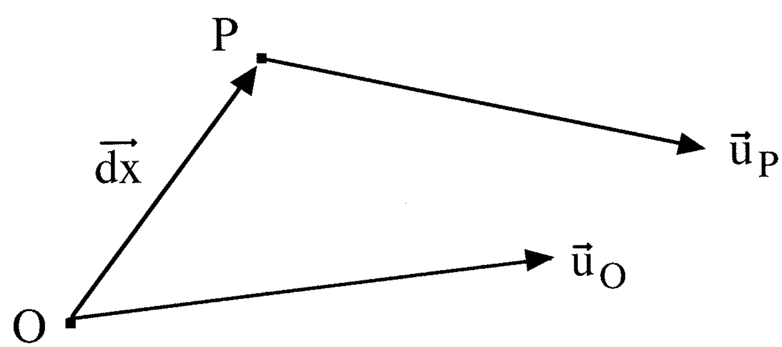
Local flow analysis

Real flows are messy. But zoom in on a small area and the flow looks a lot simpler: it's the sum of several simple flow types.

Describe flow in the vicinity of any given point O.

P is at location $\vec{x}_O + \overrightarrow{dx}$

O is at location $\vec{x}_O$



$\overrightarrow{dx}$ is directed distance from point O to a nearby point P.

Perform Taylor expansion of flow at P about point O:

$$u_i(\vec{x}_O + \overrightarrow{dx}) = u_i(\vec{x}_O) + \frac{\partial u_i}{\partial x_j}(\vec{x}_O)\, dx_j \quad (\text{h.o.t} \rightarrow 0 \text{ for } dx_j \rightarrow 0)$$

$$\underbrace{\frac{\partial u_i}{\partial x_j}}_{\text{velocity gradient tensor}} = \underbrace{\boxed{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\text{sym tensor}} + \frac{1}{2}\underbrace{\boxed{\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}}_{\text{anti-sym tensor}}$$

$$= \underset{\text{rate of strain tensor}}{e_{ij}} + \underset{\text{rotation tensor}}{\frac{1}{2} r_{ij}}$$

$$\therefore\; u_i(\vec{x}_O + \vec{dx}) = u_i(\vec{x}_O) + e_{ij}\,dx_j + \frac{1}{2} r_{ij}\,dx_j \quad \text{[e, r evaluated at O]}$$

<u>Locally</u> the velocity varies <u>linearly</u> with spatial coords.

r is an anti-sym 2nd order tensor. Has 3 distinct elements. Can always relate comps of an antisym 2nd order tensor to comps of a vector. In this case, relate comps of r to comps of vort vector $\vec{\omega}$. A 2nd order antisym tensor related to $\vec{\omega}$ is:

$$\varepsilon_{ijk}\,\omega_k = \varepsilon_{ijk}\,\varepsilon_{klm}\frac{\partial u_m}{\partial x_l} = \left(\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\right)\frac{\partial u_m}{\partial x_l}$$

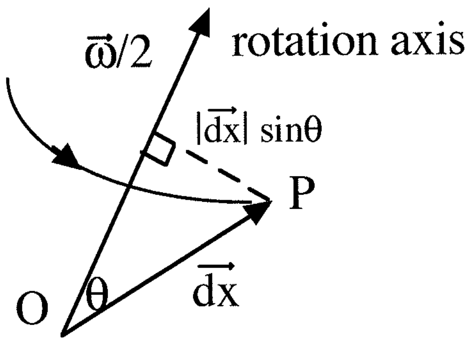
$$= \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \quad \text{this is } -r_{ij}$$

$$\therefore \quad \boxed{r_{ij} = -\varepsilon_{ijk}\,\omega_k}$$

What is meaning of $\frac{1}{2} r_{ij}\,dx_j$?

$$\frac{1}{2} r_{ij}\,dx_j = -\frac{1}{2}\varepsilon_{ijk}\,\omega_k\,dx_j = \frac{1}{2}\varepsilon_{ikj}\,\omega_k\,dx_j = \frac{1}{2}\left(\vec{\omega}\times\vec{dx}\right)_i$$

$\frac{1}{2}\vec{\omega}\times\vec{dx}$ is velocity associated w/ rigid body rotation (also known as solid body rotation) about an axis through O w/ angular velocity $\vec{\omega}/2$.



Speed at P: $\left|\frac{1}{2}\vec{\omega}\times\vec{dx}\right| = \left|\frac{\vec{\omega}}{2}\right|\left|\vec{dx}\right|\sin\theta$. It's the ang velocity $|\vec{\omega}/2|$ times distance $|\vec{dx}|\sin\theta$ $\perp$ to rotation axis. Dir^n of velocity at P: $\perp$ to $\vec{\omega}$ and $\vec{dx}$ (into page, in sense of right hand rule).

$$u_i(\vec{x}_O + \vec{dx}) = u_i(\vec{x}_O) + e_{ij}\,dx_j + \frac{1}{2} r_{ij}\,dx_j \quad \text{[e, r evaluated at O]}$$

So flow in vicinity of O is due to:

(1) translation $\quad u_i(\vec{x}_O)$

(2) deformation $\quad e_{ij}\,dx_j$
[deformation: parcels change their distances from each other]

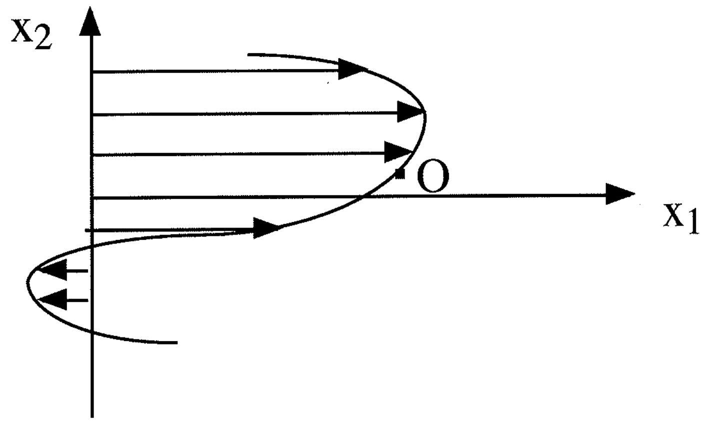
(3) rigid body rotation $\quad \frac{1}{2} r_{ij}\,dx_j$

e.g., consider local flow analysis for a parallel shear flow,

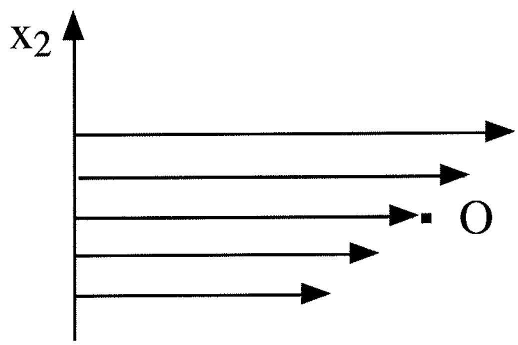
$$u = f(y), \qquad v = 0, \qquad w = 0,$$

$$[u_1 = f(x_2), \qquad u_2 = 0, \qquad u_3 = 0].$$

Flow might look like:



Zoom in on a point O of interest. For local analysis consider tiny neighborhood around O. e, r are evaluated at O [involves only 1st derivs of $\vec{u}$ evaluated at O]. Magnified view:



Evaluate comp^s of e:

$$e_{11} = \frac{\partial u_1}{\partial x_1} = 0, \quad e_{22} = \frac{\partial u_2}{\partial x_2} = 0, \quad e_{33} = \frac{\partial u_3}{\partial x_3} = 0\ ,$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \underbrace{\boxed{\frac{\partial u_2}{\partial x_1}}}_{0}\right) = \underbrace{\boxed{\frac{1}{2}\frac{df}{dx_2}}}_{\text{call it } \alpha} = \alpha, \qquad (> 0 \text{ at O})$$

$$e_{21} = e_{12} = \alpha$$

all other comp^s of e are zero.

Evaluate comp^s of r:

$$r_{11} = r_{22} = r_{33} = 0 \text{ , (since r is antisym)}$$

$$r_{12} = \frac{\partial u_1}{\partial x_2} - \underbrace{\boxed{\frac{\partial u_2}{\partial x_1}}}_{0} = \frac{df}{dx_2} = 2\alpha$$

$$r_{21} = -r_{12} = -2\alpha$$

all other comps of r are zero.

Look at u_1 near point O:

$$u_1(\vec{x}_O + \vec{dx}) = u_1(\vec{x}_O) + e_{1j}\,dx_j + \frac{1}{2} r_{1j}\,dx_j$$

$$= u_1(\vec{x}_O) + \underbrace{e_{11}}_{0} dx_1 + \underbrace{e_{12}}_{\alpha} dx_2 + \underbrace{e_{13}}_{0} dx_3$$

$$+ \frac{1}{2}(\underbrace{r_{11}}_{0} dx_1 + \underbrace{r_{12}}_{2\alpha} dx_2 + \underbrace{r_{13}}_{0} dx_3)$$

$$u_1(\vec{x}_O + \vec{dx}) = u_1(\vec{x}_O) + \underbrace{\alpha\,dx_2}_{\text{shear strain}} + \underbrace{\alpha\,dx_2}_{\text{rotation}}$$

Now look at u_2 near point O:

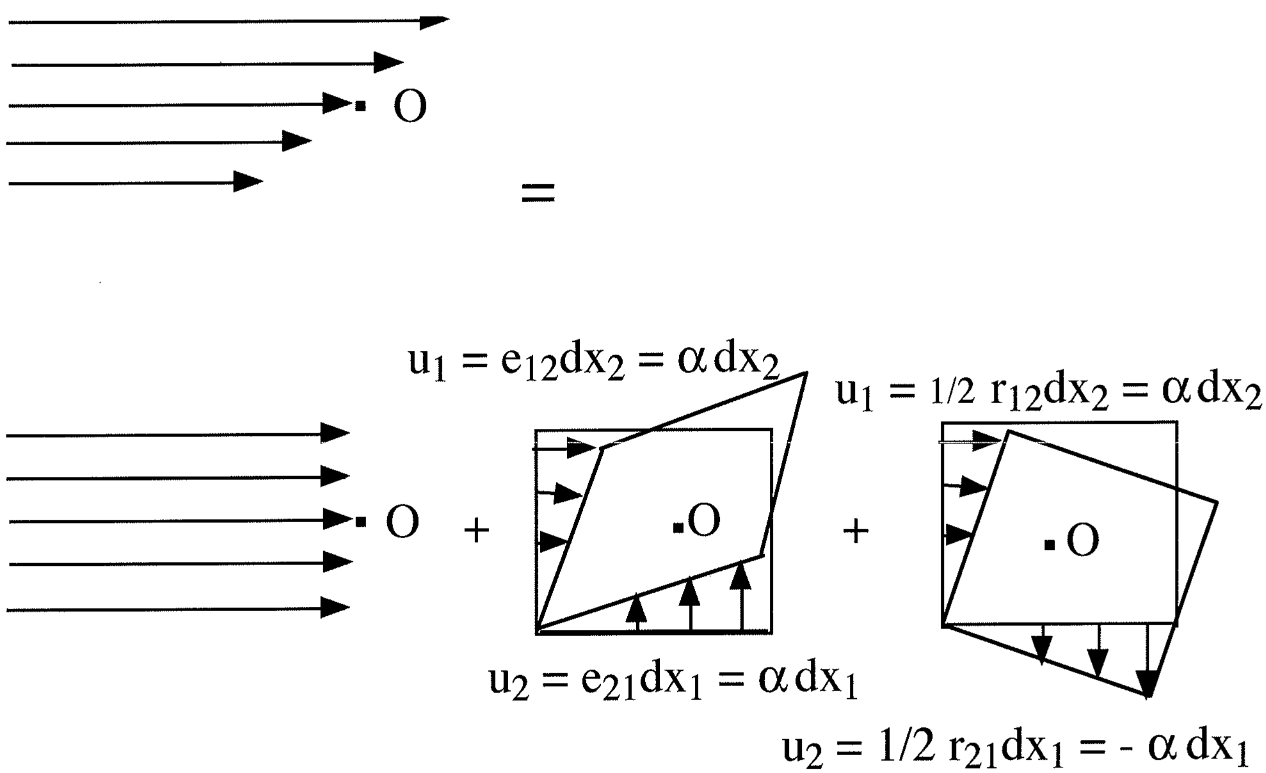
$$u_2(\vec{x}_O + \vec{dx}) = u_2(\vec{x}_O) + e_{2j}\,dx_j + \frac{1}{2} r_{2j}\,dx_j$$

$$= \underbrace{u_2(\vec{x}_O)}_{0} + \underbrace{e_{21}}_{\alpha} dx_1 + \underbrace{e_{22}}_{0} dx_2 + \underbrace{e_{23}}_{0} dx_3$$

$$+ \frac{1}{2}(\underset{-2\alpha}{r_{21}}\, dx_1 + \underset{0}{r_{22}}\, dx_2 + \underset{0}{r_{23}}\, dx_3)$$

$$u_2(\vec{x}_O + \vec{dx}) = \underset{\text{shear strain}}{\alpha\, dx_1} - \underset{\text{rotation}}{\alpha\, dx_1} = 0$$

Evaluate these formulas for u_1 and u_2 in different locations, e.g., go along x_1 axis (where $dx_2 = 0$), then go along x_2 axis (where $dx_1 = 0$). Look at the individual contributions from translation, deformation, and rotation. Find that:



translation

deformation (shear strain in this case)

rotation

[Go through first part of handout on review of linear algebra].