Dimensional Analysis

The Universe does not care about the units used to describe phenomena so it should be possible to phrase the laws of the Universe (and results following from those laws) in a dimensionless form.

It turns out that dimensionless forms are often very convenient to work with because:

(i) they collapse a given problem down to a problem involving fewer degrees of freedom, and

(ii) in the simplest problems they can lead directly to an answer. for example: $k^{-5/3}$ power law for energy in turbulent flow (inertial sub-range). Another example: in the decay of a viscous line vortex dimensional analysis reduces the pde to an ode which can then be solved analytically.

Perhaps the most important theorem/tool in dimensional analysis Buckingham's Pi Theorem.

Some good references on it:

Bridgman, P.W., 1922: Dimensional analysis

Birkhoff, G., 1950: Hydrodynamics: A study in Logic, Fact, and Similitude.

Langhaar, H.L., 1951: Dimensional Analysis and theory of Models.

see also Kundu's "Fluid Mechanics"

Preamble:

Consider a problem involving m parameters. Call them $q_1, q_2, \ldots, q_m$. We anticipate a relation btw them but we don't know what it is. Important note: here we use the term "parameter" most generally to mean a usual parameter (e.g. gravity g) or an independent variable (e.g. time t) or a dependent variable (e.g. a velocity component).

Anticipate that a parameter (say q_1) is related to the others by a functional relation of the generic form:

$$q_1 = f(q_2, q_3, \ldots, q_m),$$

or equivalently:

$$q_1 - f(q_2, q_3, \ldots, q_m) = 0,$$

or equivalently:

$$(1) \qquad f_1(q_1, q_2, \ldots, q_m) = 0.$$

Some of these parameters might be dimensionless (e.g. angle of a slope) but many will have dimensions (e.g. L length, M mass, T time, θ temp).

Let k = largest number of these parameters that will not combine into a non-dimensional form. k is usually (but not always) the # of dimensions that enter the problem, e.g. M, L, T are 3 dimensions (so k is probably 3).

This last cond^n insures that only linearly indep dimensions get counted. e.g. if mass and length only appear together in the combo mass/length then they're effectively just 1 dimension.

So k is usually # dimensions that enter a problem but possibly it's less.

<u>Buckingham's Pi Theorem</u> states that a relation equivalent to (1) exists for n <u>non-dimensional</u> parameters:

$$f_2(\Pi_1, \Pi_2, \ldots, \Pi_n) = 0$$

where $n = m - k$

n is the "Pi number" (number of "effective" parameters).

The Π's are called Pi groups. They are <u>non-dimensional</u> combinations of the original parameters. They are <u>not</u> unique.

The usefulness of this theorem depends on proper choice of q's. Don't omit any important parameter! On the other hand, don't include anything extraneous!

The theorem becomes clearer by example.

<u>Example 1</u>: At $t=0$ a penny falls from rest from a height $z=0$ in a gravity field g. Find its height z as a function of time t.

<u>Step 1: List all relevant parameters</u> (reg parameters, indep variables, and dep variable).

Ok, here they are: z, t, g So $m=3$

Clearly we are excluding a number of possible complications such as wind resistance and Coriolis force.

Step 2: Check dimensions

Use [ ] symbol, a dimension extractor

$[g] = L/T^2$

$[z] = L$

$[\tau] = T$ (sometimes "T" will be used for the dimension of temperature but it should be clear from the context).

$\therefore$ 2 dimension, L and T

$\therefore$ k is likely 2. Is it less? Look at all possible combinations of z, τ (power-laws multiplied together) → they are all dimensional. (same thing with combos of g, z and combos of g, τ). So, yes, $k=2$.

$\therefore \quad n = m - k = 3 - 2 = 1$

So pi number is 1.

So $f_1(g, z, \tau) = 0$ can be put in an equivalent form:

$f_2(\Pi_1) = 0$ where Π_1 is a non-dimensional combination of z, g, τ.

Calculate Π_1:

$\Pi_1 = z g^a \tau^b$ where a and b are exponents. We can take any one parameter (e.g. z) to have exponent 1.

$$[\Pi_1] = [z g^a \tau^b] = L\left(\frac{L}{T^2}\right)^a T^b$$
$$= L^{1+a} T^{b-2a}$$

Since Π is non-dimensional must have:

$1 + a = 0$ and $b - 2a = 0$

$\therefore$ $a = -1$ and $b = 2a = -2$

$$\therefore \Pi_1 = \frac{z}{g\tau^2}$$

Digression: If we didn't take exponent of z to be 1 we could try:
$$\Pi_1 = z^c g^a \tau^b$$
$$\therefore [\Pi_1] = [z^c g^a \tau^b] = L^c \left(\frac{L}{T^2}\right)^a T^b$$
$$= L^{c+a} T^{b-2a}$$

$\therefore$ $c + a = 0$ and $b - 2a = 0$

2 eq^ns in 3 unknowns. It's underdetermined. Take any exponent to be whatever you want (so the Π is not unique). If we take $c = 1$ we get the same Π_1 as above. Other choices could lead to:

$\Pi_1 = \frac{g\tau^2}{z}$ or $\sqrt{g/z}\,\tau$ or ...

So the Pi Theorem tells us that:

$f_2\left(\frac{z}{g\tau^2}\right) = 0$ Take the "inverse function"

$$\frac{z}{g\tau^2} = f_2^{-1}(0) = \text{some const}$$

$\therefore$ $\boxed{z = Cg\tau^2}$ where C is some const.

Can run one experiment to determine what the C is.

Note: we got the above relation by considering dimensions (only). Let's compare this result with what we get from Newton's 2nd Law:

$F = ma$

$\therefore$ $-mg = ma$ (the only force we've considering is gravity)

$\therefore$ $a = -g$

$\therefore$ $\frac{d^2 z}{d\tau^2} = -g$ integrate w.r.t. time

$\frac{dz}{dt} = -g\tau + A$ where A is a const of integration. Integrate once more.

$z = -g\frac{\tau^2}{2} + A\tau + B$

Apply the initial data: penny is initially at $z=0$ (so $z(0)=0$). This means $B=0$. Also, the penny is initially at rest (so $\left.\frac{dz}{dt}\right|_{\tau=0} = 0$). This means $A=0$.

$\therefore$ Newton's 2nd Law gives us:

$\boxed{z = -\frac{1}{2}g\tau^2}$

Comparing this with $z = Cg\tau^2$ we see that $C = -\frac{1}{2}$.

So, in this case, dimensional analysis gave us everything that Newton's 2nd Law did → except Newton's Law gave us the value of C.

Example 2

Consider the net drag force D exerted by the wind on a cube-shaped house of length (and width and height) d. We anticipate that wind speed U, air density ρ, and dynamic viscosity coefficient μ are relevant. So we expect:

$$D = f(d, U, \rho, \mu)$$

Can do laboratory (wind tunnel) experiments or numerical model experiments to find out what this function is but note that it's a function of 4 parameters so there are lots of experiments that need to be done. If each parameter is varied 10 times independently of the others that entails $10 \times 10 \times 10 \times 10 = 10^4$ experiments. Or use <u>dimensional analysis</u> to simplify the task!

D, d, U, ρ, μ are 5 parameters ($m = 5$)

dimensions: $[d] = L$

$[U] = L/T$

$[\rho] = M/L^3$

$[\mu] = \dfrac{M}{LT}$ → get it by considering $\rho \dfrac{d\vec{u}}{dt} \sim \boxed{\mu} \nabla^2 \vec{u}$

$[D] = \dfrac{ML}{T^2}$ → $[\text{force}] = [\text{mass} \times \text{accel}] = ML/T^2$

$\therefore$ 3 dimensions: L, M, and T

$\therefore$ k is likely 3. [Could it be just 2? No, can verify that we can find a combo of 3 parameters that won't combine into a non-dimensional form, e.g. d, U, ρ]

$\therefore$ $n = m - k = 5 - 3 = 2$ So pi number is 2

$\therefore$ There exists a relation of the form:

$$f_2(\Pi_1, \Pi_2) = 0$$

or equivalently:

$$\Pi_1 = f(\Pi_2)$$

Need to find these 2 pi groups (not unique). Want D to be in one of them (Π_1) but not the other so we get a clear solution for D instead of an implicit relation connecting D to itself.

So take $\Pi_1 = D\, d^a U^b \rho^c \mu^e$

$$[\Pi_1] = \frac{ML}{T^2} L^a \left(\frac{L}{T}\right)^b \left(\frac{M}{L^3}\right)^c \left(\frac{M}{LT}\right)^e$$

$$= M^{1+c+e} L^{1+a+b-3c-e} T^{-2-b-e}$$

Since Π_1 is non-dim, must have:

$$\left.\begin{aligned} &1 + c + e = 0 \\ &1 + a + b - 3c - e = 0 \\ &-2 - b - e = 0 \end{aligned}\right\}$$

3 eq^{ns} for 4 unknowns. Underdetermined. May as well set $e = 0$, i.e. don't include μ in Π_1 $\rightarrow$ but then we must include it in Π_2. Every important parameter must be included somewhere in the problem.

So, with $e = 0$ we get:

$1 + c = 0 \rightarrow \boxed{c = -1}$

$-2 - b = 0 \rightarrow \boxed{b = -2}$

$1 + \underset{-2}{a + b} - 3\underset{-1}{c} = 0 \rightarrow 1 + a - 2 + 3 = 0 \rightarrow \boxed{a = -2}$

$$\therefore \quad \Pi_1 = \frac{D}{\rho U^2 d^2} \qquad \text{[drag coefficient]}$$

Similarly we can find $\Pi_2 = \dfrac{\rho U d}{\mu}$ [Reynolds number]

So Pi Theorem tells us that:

$$\frac{D}{\rho U^2 d^2} = F\left(\frac{\rho U d}{\mu}\right)$$

So, normalized D is a function of only one non-dimensional parameter. Much simpler than considering D to be a function of 4 parameters. Just run 1 set of experiments to determine how $\frac{D}{\rho U^2 d}$ varies with Reynolds number.