Katabatic flow

- also known as mountain/valley wind or glacier wind

- it's a shallow boundary layer wind that develops over <u>sloping terrain</u> when the surface is cooled.

- common at night in mountainous terrain

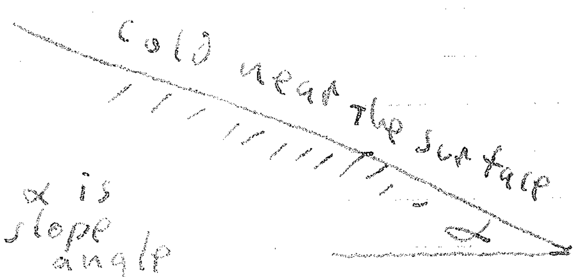
- common over ice sheets of Greenland and Antarctica (where katabatic winds can reach hurricane force).

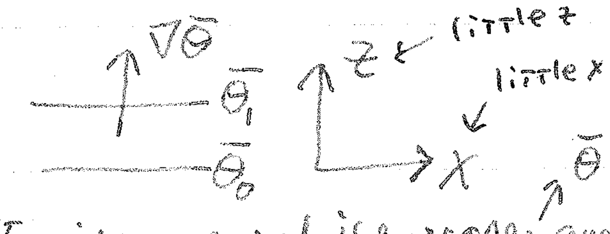
- can also find weak katabatic flows over gently sloping terrain just about anywhere if you look for it at the right time (cooled surface) and if it's not disguised by synoptic-scale flows. Often it's weak (~ 1 m/s or less)

- Generally pretty shallow (a few 100s of meters to the jet maximum over large glaciers). Maybe only 1 or 2 m to jet maximum over slopes of short hills.

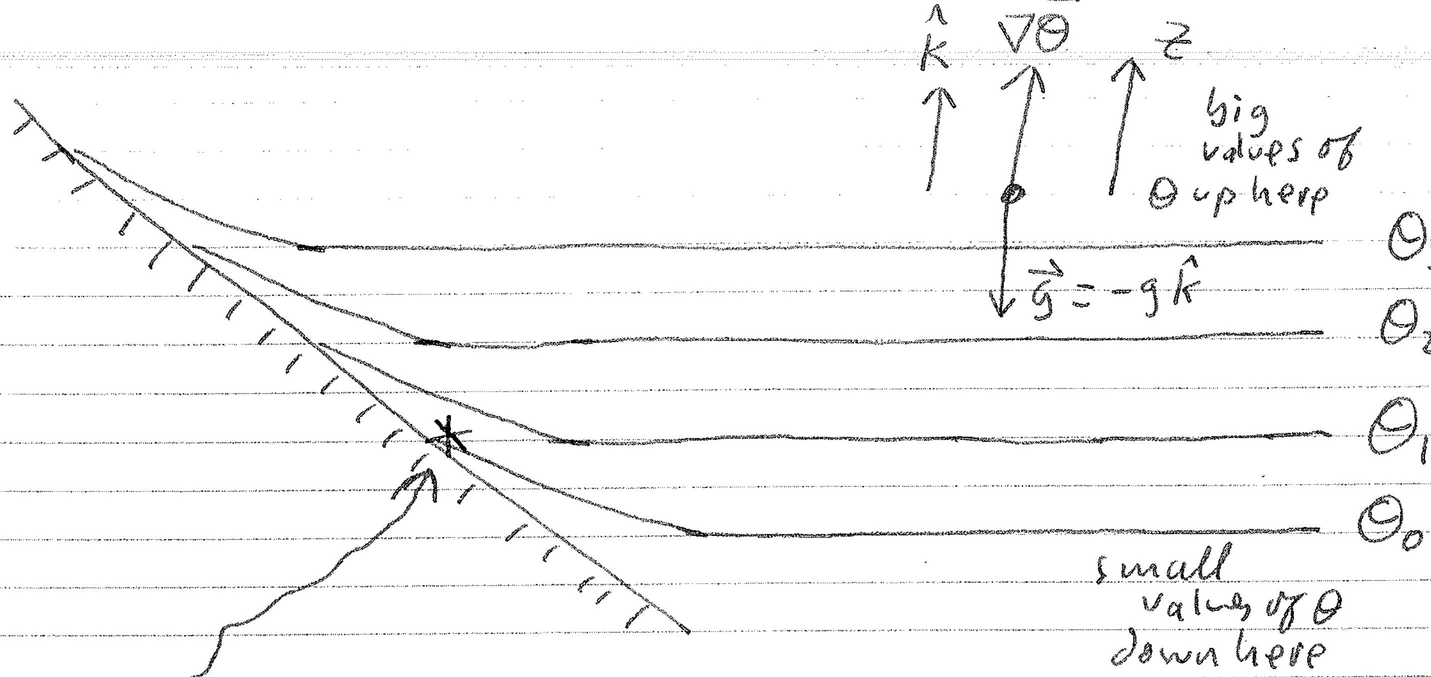
- Hard to model accurately with Numerical Prediction Models because of high resolution needed in the vertical (shallow jet) and strong static stability (tricks up turbulence parameterizations).

What induces the katabatic flow? Think of night-time ABL picture, but <u>tilted</u>:





Environmental isentropes are horizontal in free atmosphere. What do ~~they~~ θ do near the cooled surface?



Consider an air parcel near the cooled surface (marked by an $*$ above). It has a potential temperature θ_0. But at the same true altitude z as that parcel, out in the free atmosphere, the potential temperature is θ_1. And $\theta_1 > \theta_0$.

<u>So, a potential temperature difference is set up between the cold air at the surface and the warmer environmental air at the same elevation!</u>

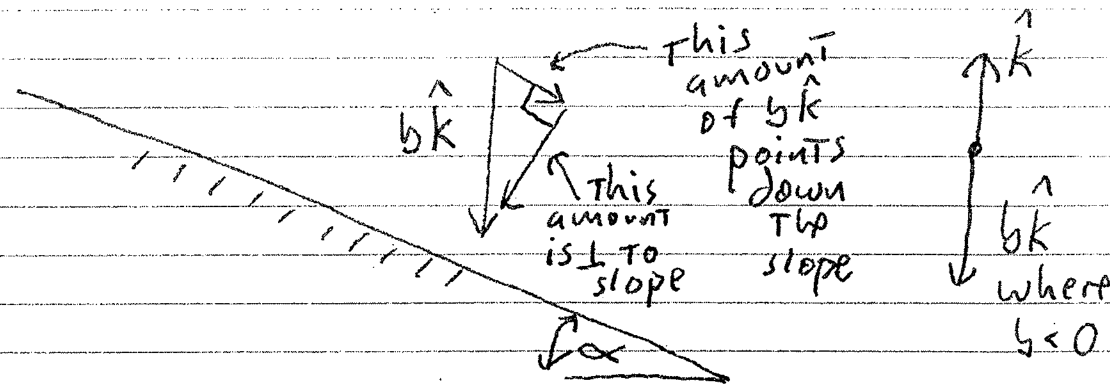
Associated with this pot temp dif is a negative value of buoyancy: $b \equiv \frac{g}{\theta_r}\left(\theta - \bar{\theta}(z)\right)$

θ_r is a constant reference potential temp

$\bar{\theta}(z)$ is height-dependent environmental pot temp
$\bar{\theta}$ ↑ w/ height ($\frac{d\bar{\theta}}{dz} > 0$)

θ is potential temp

The buoyancy force vector points <u>downward</u> (same direction as $\vec{g}$). But <u>downward</u> means partially <u>downslope</u> and partially in <u>slope-normal</u> direction.

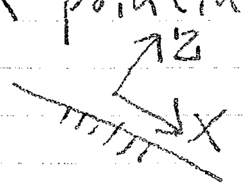


The part of the buoyancy force that projects down the slope induces a flow down the slope. This is <u>katabatic flow</u>.

<u>Prandtl (1942) model of katabatic flow.</u>

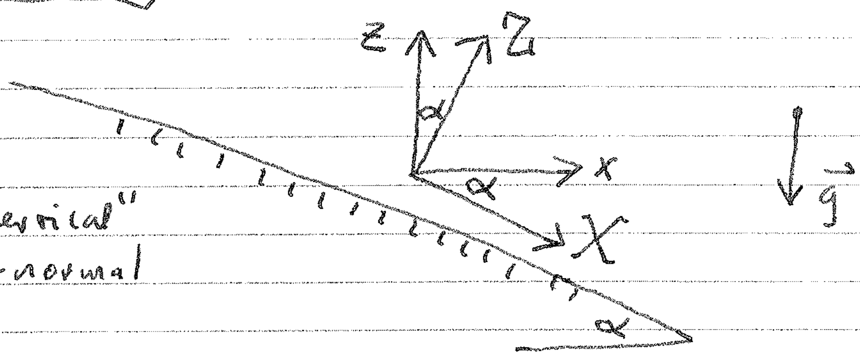
- Conceptually simple but captures the essential physics.
- Is an exact solution of the Boussinesq form of Navier-Stokes eq^{ns} and thermodynamic energy and mass conservation.
- Prandtl considered <u>steady state</u> katabatic flow along a <u>planar slope</u> (α = const) in a <u>stably-stratified environment</u> with <u>constant N</u> (so environmental pot temp increases <u>linearly with z</u>).
- slope goes on forever (no edge effects).
- Eddy viscosity ν included, but treated as constant.
- Eddy Diffusivity K " " " "
- No synoptic-scale forcings (no imposed p.g.f.)
- Coriolis force not included (so no rotation effects).

- Work in a slope-following coord system with X pointing down the slope and Z pointing ⊥ to slope. Y points across the slope (into page).



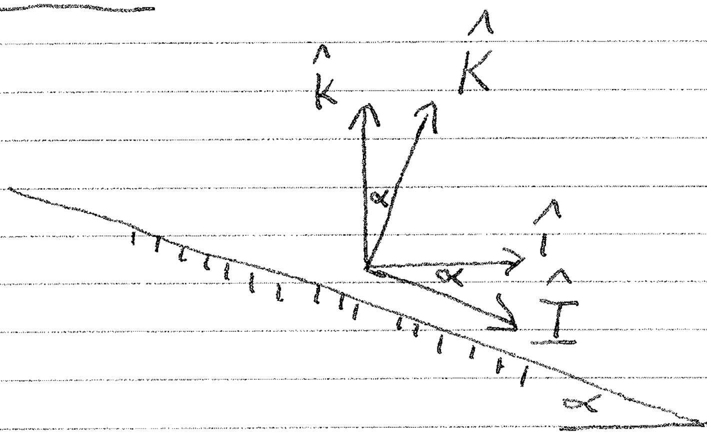
- Assume no Y-dependence in forcing so no Y-dependence in flow variables.

Original Cartesian coordinates (x, z) and new
<u>slope-following</u> Cartesian coordinates (X, Z):

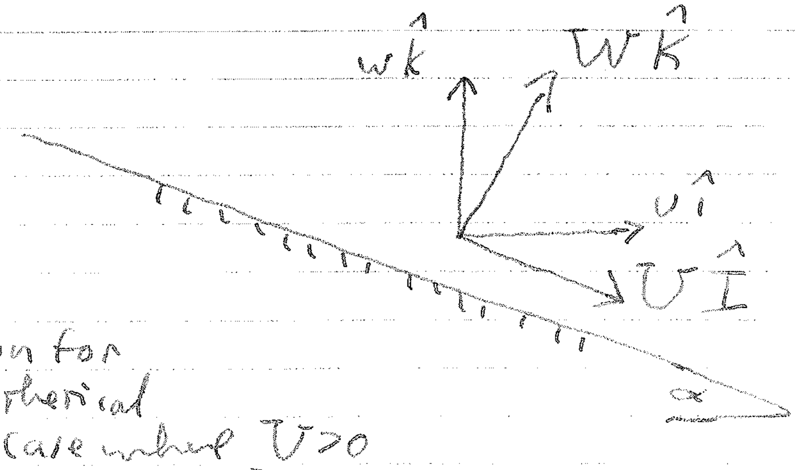


z is "true vertical"
Z is slope-normal

<u>unit vectors</u>:



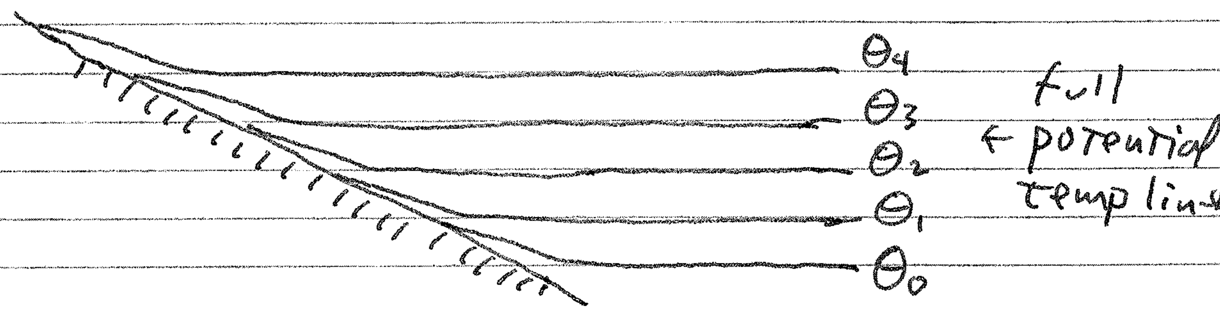
<u>velocity components</u>:

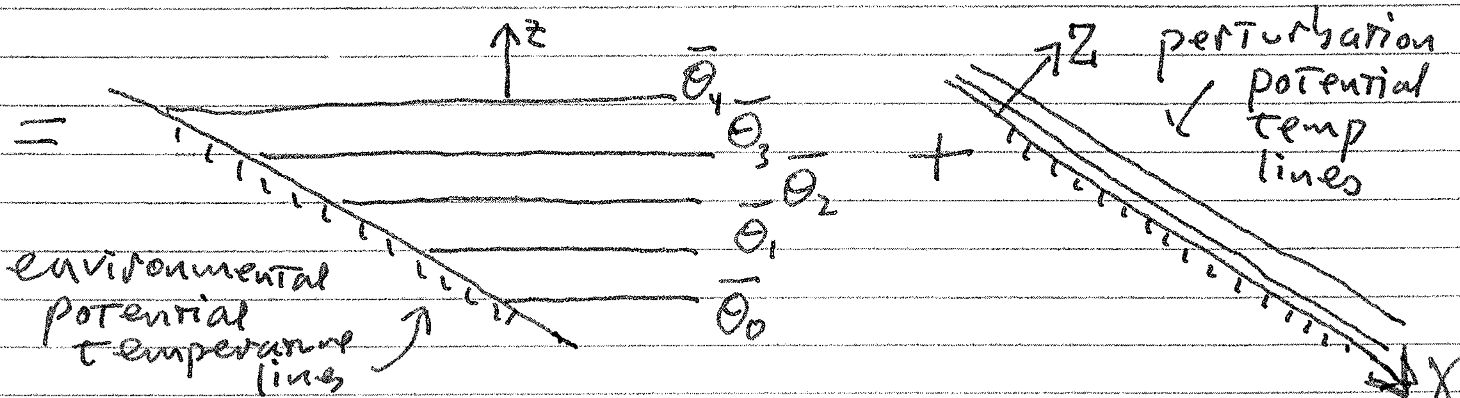


drawn for
hypothetical
case where $U > 0$
$W > 0$
$u > 0$
$w > 0$

- Radiation ignored in flow domain. It affects problem only through a parameterized role in producing a cooled lower surface.
- Assume flow is driven by a constant (in X and t) surface buoyancy.

Given these restrictions you might think the problem is 2-dimensional (X - Z) but it actually reduces to a 1-D (Z) problem. Look:





So full potential temperature varies with X and Z but perturbation potential temp varies only with Z. Anticipate that this allows us to find a katabatic flow solution ~~the~~ in which all ~~variables~~ dependent variables are independent of X (so only f^n of Z).

If such a sol^n exists then downslope wind component U is indep of X so $\frac{\partial U}{\partial X} = 0$. But then from mass conservation (incompressibility $cond^n$), must have $\frac{\partial W}{\partial Z} = 0$ where W is slope normal velocity comp.

Integrate w.r.t. Z, get $W(X,Z) = f(X)$ (function of integration). But in view of the impermeability $cond^n$ ($W=0$ at slope), must have $f(X) = 0$

$\therefore$ $W = 0$ everywhere.

So we've working with $U = U(Z)$, $W = 0$ (everywhere),
$\theta = \overline{\theta}(z) + \theta'(Z)$.
↑ little z

Look at thermo energy eq^n:

$$\boxed{\frac{D\theta}{Dt} = \kappa \nabla^2 \theta}$$

expect diffusion to be important since that's how the cold surface cools the air adjacent to it.

$$\underbrace{\frac{\partial \theta}{\partial t}}_{\substack{0 \\ \text{in steady-} \\ \text{state}}} + \vec{u}\cdot\nabla\theta = \kappa \nabla^2 \theta$$

$$\therefore \quad \underbrace{\vec{u}\cdot\nabla\theta'}_{1} + \underbrace{\vec{u}\cdot\nabla\overline{\theta}}_{2} = \underbrace{\kappa\nabla^2\theta'}_{3} + \underbrace{\kappa\nabla^2\overline{\theta}}_{4}$$

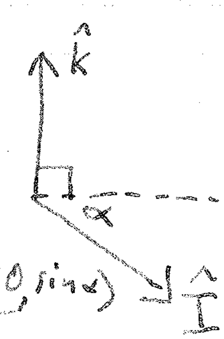
Term 1: $\vec{u}\cdot\nabla\theta' = U\hat{I}\cdot\left(\underbrace{\frac{\partial\theta'}{\partial X}}_{\substack{0 \text{ since } \theta' \text{ is} \\ \text{indep of } X}}\hat{I} + \frac{\partial\theta'}{\partial Z}\hat{K}\right)$

$$= U\frac{\partial\theta'}{\partial Z}\hat{I}\cdot\hat{K}$$

$$= 0 \quad \text{since } \hat{I} \perp \hat{K}$$

Term 2: $\vec{u}\cdot\nabla\overline{\theta} = U\hat{I}\cdot\left(\frac{d\overline{\theta}}{dz}\hat{k}\right)$ little z, little $\hat{k}$

$$= U\frac{d\overline{\theta}}{dz}\hat{I}\cdot\hat{k}$$



$$= U\frac{d\overline{\theta}}{dz}\cos(90+\alpha)$$

$$= U\frac{d\overline{\theta}}{dz}(\underbrace{\cos 90}_{0}\cos\alpha - \underbrace{\sin 90}_{1}\sin\alpha)$$

$$= -U\frac{d\overline{\theta}}{dz}\sin\alpha$$

term 3: $\kappa \nabla^2 \theta' = \kappa \left(\underbrace{\frac{\partial^2 \theta'}{\partial X^2}}_{0 \text{ since } \theta' \text{ is indep of } X} + \frac{\partial^2 \theta'}{\partial Z_1^2} \right)$

$= \kappa \frac{d^2 \theta'}{d Z_1^2}$

term 4: $\kappa \nabla^2 \bar{\theta} = \kappa \left(\frac{\partial^2 \bar{\theta}}{\partial X^2} + \frac{\partial^2 \bar{\theta}}{\partial Z_1^2} \right)$ slope-following coords

and it's also $= \kappa \left(\frac{\partial^2 \bar{\theta}}{\partial x^2} + \frac{\partial^2 \bar{\theta}}{\partial z^2} \right)$ original coords

[$\frac{\partial^2 \bar{\theta}}{\partial x^2} \neq \frac{\partial^2 \bar{\theta}}{\partial X^2}$ and $\frac{\partial^2 \bar{\theta}}{\partial z^2} \neq \frac{\partial^2 \bar{\theta}}{\partial Z_1^2}$ but their sums are equal]

This second expression is easier to work with since $\bar{\theta} = \bar{\theta}(z)$.

So: $\frac{\partial^2 \bar{\theta}}{\partial x^2} = 0$ (no x-dependence)

and since $\bar{\theta}$ varies linearly with z $(\bar{\theta} = m + nz)$,

~~so~~ $\frac{\partial \bar{\theta}}{\partial z}$ is constant so $\frac{\partial^2 \bar{\theta}}{\partial z^2} = 0$!

$\therefore \kappa \nabla^2 \bar{\theta} = 0$

Put 'em all together (put on right hand side), get:

$$\boxed{0 = U \frac{d\bar{\theta}}{dz} \sin\alpha + \kappa \frac{d^2 \theta'}{d Z_1^2}}$$ Thermo eqn

So, ~~an energy~~ it's a balance btw downslope advection of environmental potential temperature which tries to warm things up, and vertical diffusion of θ' which tries to cool.

mult prev eq^n by $\frac{g}{\theta_r}$ ← const ref pot temp

$$0 = V \frac{g}{\theta_r}\frac{d\bar{\theta}}{dz}\sin\alpha + \kappa \frac{d^2}{dZ^2}\left(\frac{g}{\theta_r}\theta'\right)$$

note that $N^2 = \frac{g}{\theta_r}\frac{d\bar{\theta}}{dz}$ and $b \equiv \frac{g}{\theta_r}(\theta - \bar{\theta}(z)) = \frac{g}{\theta_r}\theta'$

So thermo eq^n becomes:

$$\boxed{0 = V N^2 \sin\alpha + \kappa \frac{d^2 b}{dZ^2}}$$

Now look at Boussinesq eq^n of motion (vector form):

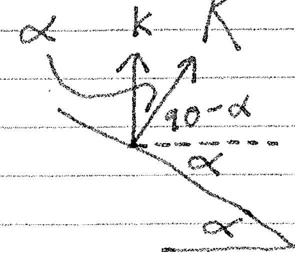
$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' + b\hat{k} + \nu\nabla^2\vec{u}$$

The Y-comp of it is $\underbrace{\frac{Dv}{Dt}}_{0} = -\frac{1}{\rho_0}\underbrace{\frac{\partial p'}{\partial Y}}_{0} + \nu\underbrace{\nabla^2 v}_{0}$

$\boxed{\text{So } 0 = 0 \text{ okay!}}$

The Z-comp of it is

$$\underbrace{\frac{DW}{Dt}}_{0} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial Z} + b\underbrace{\hat{k}\cdot\hat{K}}_{\cos\alpha} + \nu\nabla^2 W$$



0 since W=0 for all X, Y, Z, t

$\therefore$ $\boxed{\frac{\partial p'}{\partial Z} = \rho_0 b\cos\alpha}$ Z-comp eq^n of motion

So $\underbrace{\frac{\partial}{\partial X}\frac{\partial p'}{\partial Z_1}}_{\text{reverse order of differentiation}} - \rho_0 \cos\alpha \underbrace{\frac{\partial b}{\partial X}}_{0} = 0$

$$\frac{\partial}{\partial Z_1}\frac{\partial p'}{\partial X} = 0 \quad \text{int w.r.t. } Z_1:$$

$$\frac{\partial p'}{\partial X}(X, Z) = G(X) \leftarrow \text{function of integration}$$

Get $G(X)$ by evaluating above expression far above the slope. Since there's no synoptic-scale p.g.f., $G(X) = 0$

$\therefore \frac{\partial p'}{\partial X} = 0$ <u>everywhere</u>.

Now examine the X-comp eq^n of motion:

$$\frac{DU}{D\tau} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial X} + b\,\hat{k}\cdot\hat{I} + \nu\nabla^2 U$$

$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + W\frac{\partial U}{\partial Z_1}$; $\frac{\partial U}{\partial \tau} \to 0$ steady-state; $U\frac{\partial U}{\partial X} \to 0$, $V \to 0$, $\frac{\partial U}{\partial Y} \to 0$, $W \to 0$

$-\frac{1}{\rho_0}\frac{\partial p'}{\partial X}$: just showed it's 0

$\hat{k}\cdot\hat{I}$: already showed it's $-\sin\alpha$

$\nabla^2 U \to \frac{d^2 U}{dZ_1^2}$ since U is indep of X

$$\therefore \boxed{0 = -b\sin\alpha + \nu\frac{d^2 U}{dZ_1^2}} \quad X\text{-comp eq}^n \text{ of motion}$$

Says: downslope component of buoyancy force balances friction force (downslope comp of it).

So, to recap:

$$\boxed{\begin{aligned} 0 &= -b\sin\alpha + \nu\frac{d^2 U}{dZ_1^2} \quad (1)\\ 0 &= UN^2\sin\alpha + \kappa\frac{d^2 b}{dZ_1^2} \quad (2)\end{aligned}}$$

2 eq^{ns} in 2 unknowns U, b

Lets solve these eq^{ns} by rewriting as 1 eq^n in 1 unknown.

From (1): $b = \frac{\nu}{\sin\alpha} \frac{d^2 U}{d Z_1^2}$ plug this into (2):

$$0 = U N^2 \sin\alpha + \frac{K\nu}{\sin\alpha} \frac{d^4 U}{d Z_1^4}$$

$$\therefore \boxed{\frac{d^4 U}{d Z_1^4} + \left(\frac{N^2 \sin^2\alpha}{K\nu}\right) U = 0}$$

4th order linear homogeneous const coefficient ordinary differential equation (o.d.e.)

Seek sol^{ns} of the form:

$$U = e^{m Z_1} \quad (\text{times constant})$$

plug into the o.d.e., get

$$m^4 e^{m Z_1} + \frac{N^2 \sin^2\alpha}{K\nu} e^{m Z_1} = 0$$

$$\therefore m^4 = -\frac{N^2 \sin^2\alpha}{K\nu}$$

$$\therefore m^2 = \pm i \frac{N \sin\alpha}{\sqrt{K\nu}}$$

$$\therefore m = \pm\sqrt{\pm i}\, \frac{\sqrt{N \sin\alpha}}{(K\nu)^{1/4}}$$

So, we need to investigate $\sqrt{\pm i}$

/

- 2 handouts on Ksat. flow

Digression on taking roots of complex numbers

Euler's formula:

$$\boxed{e^{i\phi} = \cos\phi + i\sin\phi}$$

So $e^{i\phi+2\pi i} = e^{i(\phi+2\pi)} = \cos(\phi+2\pi) + i\sin(\phi+2\pi)$

$= \cos\phi + i\sin\phi$

$= e^{i\phi}$

So adding $2\pi i$ to the exponent changes nothing.

Similarly, $e^{i\phi+2n\pi i} = e^{i(\phi+2n\pi)}$

where n is an integer

$= \cos(\phi+2n\pi) + i\sin(\phi+2n\pi)$

$= \cos\phi + i\sin\phi$

$= e^{i\phi}$

$\therefore\ e^{i\phi+2n\pi i} = e^{i\phi}$

So adding $2n\pi i$ to exponent changes nothing. But we can use that trick to help us evaluate roots of complex numbers.

We'll ~~also~~ first need to write the complex number in complex exponential form. For i, note that

$e^{i\pi/2} = \underbrace{\cos\frac{\pi}{2}}_{0} + i\underbrace{\sin\frac{\pi}{2}}_{1} = i \quad \therefore \boxed{i = e^{i\pi/2}}$

But also:

$$i = e^{i\pi/2 + 2n\pi i}$$

so

$$i^{1/2} = e^{[i\pi/2 + 2n\pi i]\frac{1}{2}}$$

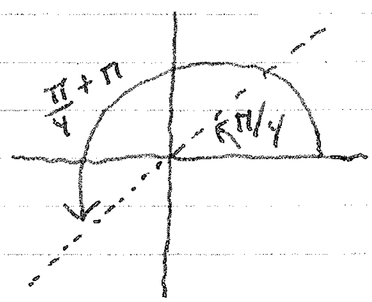
$$= e^{i(\pi/4 + n\pi)}$$

So we need to investigate the cases of dif n

For $n=0$: $i^{\frac{1}{2}} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

For $n=1$: $i^{\frac{1}{2}} = e^{i(\pi/4 + \pi)} = \cos(\frac{\pi}{4}+\pi) + i\sin(\frac{\pi}{4}+\pi)$

$$= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$



For $n=2$:

$i^{\frac{1}{2}} = e^{i(\pi/4 + 2\pi)} = e^{i\pi/4}$ <u>so same as $n=0$</u>

For $n=3$:

$$i^{\frac{1}{2}} = e^{i(\pi/4 + 3\pi)}$$

$$= e^{i(\pi/4 + \pi + 2\pi)}$$

$= e^{i(\pi/4 + \pi)}$ <u>so same as $n=1$</u>

...and so on... so there are only two roots ~~of~~ $i^{1/2}$:

$$i^{1/2} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

and $i^{1/2} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

Now evaluate $\sqrt{-i}$:

$$\sqrt{-i} = \sqrt{-1 \cdot i} = \sqrt{-1}\,\sqrt{i} = i\sqrt{i}$$

already evaluated it. Two possibilities.

$$\therefore \sqrt{-i} = i\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
$$= i\frac{1}{\sqrt{2}} + i^2\frac{1}{\sqrt{2}}$$
$$= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

or
$$\sqrt{-i} = i\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$
$$= -i\frac{1}{\sqrt{2}} - i^2\frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

End of digression.

So $m = \pm\sqrt{\pm i}\,\dfrac{\sqrt{N\sin\alpha}}{(KD)^{1/4}}$ is in $U = e^{mz}$

with 4 possible m. Label them:

$$m_1 = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{N\sin\alpha}}{(KD)^{1/4}} = \frac{(1+i)}{\delta}$$

where $\delta = \dfrac{\sqrt{2}\,(KD)^{\frac{1}{4}}}{\sqrt{N\sin\alpha}}$

δ has units of length

$$m_2 = \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{N\sin\alpha}}{(KD)^{1/4}} = \frac{(-1-i)}{\delta}$$

$$m_3 = \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{N\sin\alpha}}{(KD)^{1/4}} = \frac{(-1+i)}{\delta}$$

$$m_4 = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{N\sin\alpha}}{(KD)^{1/4}} = \frac{(1-i)}{\delta}$$

The general sol^n of our 4^{th} order o.d.e. is a linear combination of the 4 linearly independent sol^{ns} associated with these roots. So:

$$V = Ce^{m_1 z} + De^{m_2 z} + Ee^{m_3 z} + Fe^{m_4 z}$$

where C, D, E, F are constants we need to determine.

Consider behavior of each of these terms far above slope:

$$\lim_{z\to\infty} e^{m_1 z} = \lim_{z\to\infty} e^{(1+i)z/\delta} = \lim_{z\to\infty} \underbrace{e^{z/\delta}}_{\text{This part blows up!}} \underbrace{e^{iz/\delta}}_{\text{This part wiggles}} \to \infty$$

It blows up as $z\to\infty$ which is unphysical. Don't let it contaminate the sol^n! So take $\boxed{C=0}$.

$$\lim_{z\to\infty} e^{m_2 z} = \lim_{z\to\infty} \underbrace{e^{-z/\delta}}_{\text{This part dies!}} \underbrace{e^{-iz/\delta}}_{\text{This part wiggles}} \to 0$$

Okay that term behaves nicely. We'll let it survive. Next up:

$$\lim_{z\to\infty} e^{m_3 z} = \lim_{z\to\infty} e^{-z/\delta} e^{iz/\delta} \to 0$$ Okay, you can live.

$$\lim_{z\to\infty} e^{m_4 z} = \lim_{z\to\infty} e^{z/\delta} e^{-iz/\delta} \to \infty$$ Not good...not good...

So need to set $\boxed{F=0}$.

So we're left with: $V = De^{m_2 z} + Ee^{m_3 z}$

Now impose no-slip cond^n: $V(0)=0$

$$\therefore\ 0 = D + E$$

$$\therefore\ \boxed{E = -D}$$

The remaining b.c. concerns the buoyancy on the slope ($Z=0$). Can specify buoyancy or buoyancy flux (slope-normal derivative of buoyancy). Let's work with <u>specified buoyancy</u>!

$$b(0) = b_0$$

Need to translate this into a cond^n on V. Recall that (1) is:

$$0 = -b\sin\alpha + \nu \frac{d^2V}{dZ^2}$$

$$\therefore \quad \left.\frac{d^2V}{dZ^2}\right|_{Z=0} = \left.\frac{b\sin\alpha}{\nu}\right|_{Z=0}$$

$$\therefore \quad \left[\frac{d^2}{dZ^2}\left(De^{m_2Z} - De^{m_3Z}\right)\right]\Big|_{Z=0} = \frac{b_0\sin\alpha}{\nu}$$

$$\left(Dm_2^2 e^{m_2Z} - Dm_3^2 e^{m_3Z}\right)\Big|_{Z=0} = \frac{b_0\sin\alpha}{\nu}$$

$$\therefore \quad D\left(m_2^2 - m_3^2\right) = \frac{b_0\sin\alpha}{\nu}$$

$$D = \frac{b_0\sin\alpha}{\nu\left(m_2^2 - m_3^2\right)}$$

$$m_2^2 - m_3^2 = \frac{(-1-i)^2}{\delta^2} - \frac{(-1+i)^2}{\delta^2} = \frac{1}{\delta^2}\left((1+i)^2 - (-1+i)^2\right)$$

$$= \frac{1}{\delta^2}\left(1 + 2i + \underbrace{i^2}_{=-1} - (1 - 2i + \underbrace{i^2}_{=-1})\right)$$

$$= \frac{1}{\delta^2}(2i + 2i) = \frac{4i}{\delta^2}$$

$$\therefore \quad D = \frac{b_0\sin\alpha\,\delta^2}{\nu 4i} = \frac{b_0\sin\alpha}{\nu 4i}\left(\frac{\sqrt{2}\,(\kappa\nu)^{1/4}}{\sqrt{N\sin\alpha}}\right)^2 \cancel{= b_0\sin\alpha}$$

$$\therefore \quad D = \frac{b_0 \sin\alpha}{2 \nu i}\left(\frac{2\sqrt{\kappa\nu}}{N \sin\alpha}\right) = \frac{b_0}{N 2i}\sqrt{\frac{\kappa}{\nu}}$$

and since $U = D e^{m_2 Z} - D e^{m_3 Z}$:

$$U = \frac{b_0}{N 2i}\sqrt{\frac{\kappa}{\nu}}\left(e^{(-1-i)Z/\delta} - e^{(-1+i)Z/\delta}\right)$$

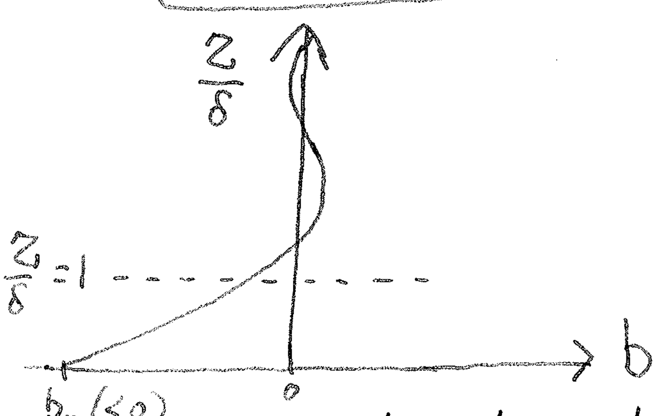
$$= \frac{b_0}{N}\sqrt{\frac{\kappa}{\nu}}\, e^{-Z/\delta}\underbrace{\left(\frac{e^{-iZ/\delta} - e^{iZ/\delta}}{2i}\right)}_{\text{all of this is } -\sin\frac{Z}{\delta}}$$

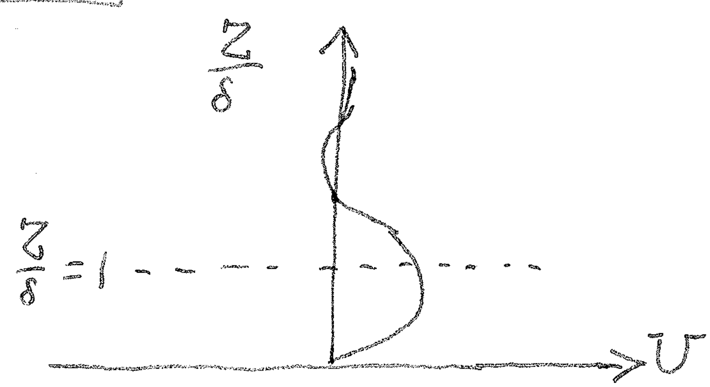
$$\therefore \quad \boxed{U = -\frac{b_0}{N}\sqrt{\frac{\kappa}{\nu}}\, e^{-Z/\delta} \sin\left(\frac{Z}{\delta}\right)}$$

To get b, return to (1) : $0 = -b\sin\alpha + \nu \frac{d^2 U}{dZ^2}$

Plug in the sol^n for U and evaluate it. Get b as:

$$\boxed{b = b_0\, e^{-Z/\delta} \cos\left(\frac{Z}{\delta}\right)}$$





→ Look at handouts of more accurately drawn b, U!

If $b_0 < 0$ then $\bar{U} > 0$ near slope $\therefore$ downslope flow

If $|b_0|$ is large then $\bar{U}$ is large.

Most of the cold surface air is beneath the level $Z = \delta$ so δ is a convenient measure of boundary layer thickness (also, roughly height of katabatic jet).

$$\delta = \frac{\sqrt{2}\,(K\nu)^{1/4}}{\sqrt{N \sin\alpha}} \quad \text{so } \delta \uparrow \text{ if turbulence level} \uparrow$$

But note that the sol^n also says: get <u>positively</u> buoyant air capping the cold <u>katabatic boundary</u> layer air. Why? Because <u>downslope wind</u> is pulling down the large-θ air; above a certain height that effect wins out over cooling. At even higher altitude that capping warm air leads to <u>buoyant upslope</u> flow (flow reversal).

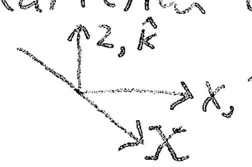
Can also interpret the katabatic flow from a vorticity dynamics perspective.

Consider the <u>vector</u> eq^n of motion for this flow (noting that we've shown that all non-linear terms are 0):

$$0 = -\frac{1}{\rho_0}\nabla p' + b\hat{k} + \nu\nabla^2\vec{u}$$

Take $\nabla \times$ (eq^n). Use $\nabla \times \nabla p' = 0$ (curl of a gradient of a scalar is 0)

Expand $\nabla \times (b\hat{k})$ into components in the <u>original</u> Cartesian coord system (<u>not</u> slope-following).



y (and $\hat{j}$) <u>point into pg</u> (y and Y the same; $\hat{j}$ and $\hat{J}$ the same)

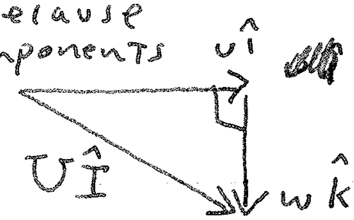
$$\nabla \times b\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & b \end{vmatrix} = \hat{i}\underbrace{\frac{\partial b}{\partial y}}_{\substack{0 \text{ since } b \text{ is} \\ \text{indep of } y\ (Y)}} - \hat{j}\frac{\partial b}{\partial x} = -\hat{j}\frac{\partial b}{\partial x}$$

$$\nabla \times \nu \nabla^2 \vec{u} = \nu \nabla^2 (\nabla \times \vec{u}) = \nu \nabla^2 \vec{\omega}$$

where $\vec{\omega} \equiv \nabla \times \vec{u}$ is vorticity vector.

$$\vec{\omega} = \nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & 0 & w \end{vmatrix} = \hat{i}\underbrace{\frac{\partial w}{\partial y}}_{0} - \hat{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) - \hat{k}\underbrace{\frac{\partial u}{\partial y}}_{0}$$

little u, w are NOT 0 because $U\hat{I}$ has components in the $\hat{i}$ and $\hat{k}$ directions: $U\hat{I}$



$$\therefore \vec{\omega} = \hat{j}\underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{\text{call it } \eta}$$

so there is j-comp vorticity.

Note: η is $\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ but it's also $\frac{\partial U}{\partial Z}$
why?

$\therefore$ vort eqn becomes:

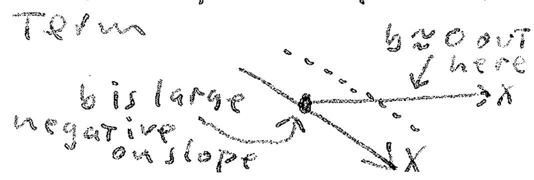
$$0 = -\hat{j}\frac{\partial b}{\partial x} + \nu \nabla^2 \eta \hat{j}$$

only j-comp vort is relevant.

Take $\hat{j}\cdot$(eqn):

$$\boxed{0 = \underbrace{-\frac{\partial b}{\partial x}}_{\substack{\text{baroclinic} \\ \text{generation} \\ \text{term}}} + \underbrace{\nu \nabla^2 \eta}_{\substack{\text{diffusion} \\ \text{of vorticity} \\ \text{term.}}}}$$

Even though b is constant all along the slope (X), it does vary in the horiz (x) direction and this generates vorticity!



→ digression on right hand rule for shear vort and curvature vort!

So $\frac{\partial b}{\partial x}$ near slope (in katabatic jet) is <u>positive</u>.

So $0 = \underbrace{-\underbrace{\frac{\partial b}{\partial x}}_{+}}$ $+ \nu \nabla^2 \eta$

so baroclinic term is <u>negative</u> near slope.

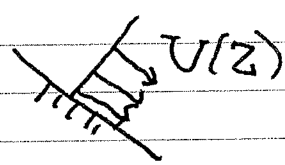
$\therefore$ <u>$\nabla^2 \eta > 0$</u> near slope

In slope following coords, since $\eta = \eta(z)$ $\nabla^2 \eta > 0$ ~~this~~ means:

$\frac{d^2 \eta}{dz^2} > 0$ near slope.

and η <u>starts out positive</u> ($\eta > 0$ at $z = 0$) due to friction (no-slip cond^n).

no friction:

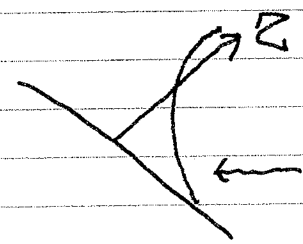


with friction:

so $\frac{dU}{dz} > 0$ at slope

$\therefore \eta > 0$ at slope

So <u>η-profile</u> near slope:



So <u>large positive η</u> generated at surface <u>diffuses upward</u>.

At any point ~~at~~ near the slope (in jet), tendency to generate <u>negative vorticity</u> <u>baroclinically</u> is <u>opposed</u> by <u>diffusion</u> of vorticity. Get a stalemate (steady-state).