→ handout: Katabatic paper by Shapiro + Fedorovich (2008)

Our discussion of Prandtl sol^n for steady-state laminar 1-D katabatic flow down a planar slope is done. Now see what happens if we relax some of the restrictions.

Unsteady Prandtl flow Consider same Prandtl scenario as above, but this time examine the unsteady problem, i.e. the development of the flow starting from rest. At $t = 0$ suddenly turn on the cold surface. Can solve the problem analytically with Laplace Transforms. In the interest of time, lets just cut to the numerical sol^n (direct numerical simulation)

→ See Fig 2 of Shapiro + Fedorovich (2008). Everything is non-dimensional. B is buoyancy, U is downslope velocity, T is time. Fig shows gravity oscillation that slowly damps with time. Eventually a steady-state is reached (= to the steady-state Prandtl sol^n), but this occurs at times later than shown on the Figure.

Turbulent unsteady Prandtl flow For large enough buoyancy forcing the flow becomes turbulent.

→ See Fig 4 of Shapiro + Fedorovich (2008). Same quantities as in Fig 2. Note change of scale (Z-axis labels in Fig 4 versus Fig 2): the boundary layer in Fig 4 (turbulent) is much larger than boundary layer in Fig 2 (laminar) case. Still get gravity oscillations but they do NOT damp out with time (not shown in plot)! Solution approaches a statistically stationary state at long times. Really, a periodic (oscillatory) stationary state. Note that the laminar and turbulent flows in Figs. 2 and 4 are qualitatively similar for the times shown (later on the oscillations die in laminar case but not turbulent case).

Steady-state Prandtl-like flow w/ Coriolis force

So: laminar, 1-D (Z), planar slope, steady-state, Coriolis force included.

So set up same Prandtl model eq^{ns} as before but include <u>Coriolis terms</u> + a <u>cross-slope</u> eq^n of motion.

<u>Thermo:</u> $$0 = U N^2 \sin\alpha + \kappa \frac{d^2 b}{dZ^2}$$

<u>downslope eq^n of motion:</u> $$0 = -b\sin\alpha + fV + \nu \frac{d^2 U}{dZ^2}$$

<u>cross-slope eq^n of motion:</u> $$0 = -fU + \nu \frac{d^2 V}{dZ^2}$$

where f is Coriolis parameter.

Multiply thermo eq^n by f, and multiply cross-slope eq^n by $N^2\sin\alpha$, and add them up. This eliminates the U term:

$$0 = fUN^2\sin\alpha + f\kappa\frac{d^2 b}{dZ^2} - N^2 \sin\alpha\, fU + N^2\sin\alpha\,\nu\frac{d^2V}{dZ^2}$$

cancel

$$\therefore \quad f\kappa \frac{d^2 b}{dZ^2} + N^2 \sin\alpha\,\nu \frac{d^2 V}{dZ^2} = 0$$

integrate w.r.t. Z:

$$f\kappa \frac{db}{dZ} + N^2\sin\alpha\,\nu\frac{dV}{dZ} = C, \text{ a constant}$$

Assuming this is a pure katabatic flow (i.e. no synoptic-influences to affect it), we ~~[illegible]~~ want b and V to vanish as $Z \to \infty$ so db/dz, dV/dZ should go to 0 as $Z \to \infty$

So C must be 0. [Note: having $dB/dz \to 0$, $dV/dz \to 0$ or $z \to \infty$ is consistent with B and V going to 0 or *any* const value]

$$\therefore \; f\kappa \frac{db}{dz} + N^2 \sin\alpha \, \nu \frac{dV}{dz} = 0$$

int again:

$$f\kappa b + N^2 \sin\alpha \, \nu V = D, \text{ a const}$$

"dee", not 0

On the slope ($z=0$) impose the no-slip cond^n ($V(0)=0$). And consider case where the slope buoyancy is specified, as in the original Prandtl scenario. So $b(0) = b_0$.

So, evaluating prev eq^n at $z=0$, get:

$$f\kappa b_0 + N^2 \sin\alpha \, \nu \cdot 0 = D$$

$$\therefore \; D = f\kappa b_0$$

$$\therefore \; f\kappa b + N^2 \sin\alpha \, \nu V = f\kappa b_0$$

Write down this formula for $z=\infty$ ($b(\infty)=b_\infty$, $V(\infty)=V_\infty$):

$$(\star) \quad \boxed{f\kappa b_\infty + N^2 \sin\alpha \, \nu V_\infty = f\kappa b_0}$$

well, we wanted b_∞ and V_∞ to both be 0 (disturbance dies out as $z \to \infty$) but ($\star$) tells us that's not possible. [If $b_\infty = 0$ and $V_\infty = 0$ then ($\star$) would yield $0 + 0 = f\kappa b_0$, so $b_0 = 0$ so no forcing, no flow] Not good... not good.

Now evaluate <u>downslope eq^n of motion</u> as $z \to \infty$, assuming $U \to 0$ up there (so $\lim_{z\to\infty} \frac{d^2U}{dz^2} \to 0$).

$\therefore$ (✡✡) $\quad -b_\infty \sin\alpha \quad + f V_\infty = 0$

(✡) and (✡✡) are 2 eq^{ns} for 2 unknowns. Solve them for b_∞ and V_∞ as:

$$b_\infty = b_0\left(\frac{f^2}{f^2 + \frac{\nu}{\kappa}N^2\sin^2\alpha}\right) \qquad V_\infty = \frac{b_0 \sin\alpha}{f}\left(\frac{f^2}{f^2+\frac{\nu}{\kappa}N^2\sin^2\alpha}\right)$$

Not good... not good... neither b_∞ nor V_∞ are 0.

So b and V approach non-zero constant values as $Z \to \infty$.

These relations show the Coriolis version of the Prandtl problem does not have a solution with the boundary cond^{ns} we wanted to impose at ∞.

To help investigate this further, consider the following unsteady problem: [Note my bad notation. In following, b and B mean same thing]

Unsteady Prandtl-like flow with Coriolis force

Was solved analytically by Gutman + Malbakhov (1964), but we're just going to look at the numerical sol^n. Consider flow that starts from rest. At $t=0$ turn on the surface cooling.

→ Fig. 3 of Shapiro + Fedorovich (2008) shows the flow evolution. U-comp oscillates in time, and eventually the oscillation dies out (not shown). Note that U does NOT have a tendency to grow upward in time. Get katabatic jet. In contrast, B and V both oscillate and also grow upward with time! The B and V boundary layers get taller and taller. The B, V boundary layers leak upward. Continues to happen for times later than shown in figure. As $t \to \infty$ and $z \to \infty$, B, V approach the values given above for the steady state.

Turbulent unsteady Prandtl-like flow with Coriolis

What if we consider same flow as before (so with Coriolis; flow starts up from rest) but this time consider sfc forcing large enough that flow becomes turbulent. In the case of turbulent flow, will B and V be well-behaved, i.e. not continue to grow upward w/ time?

→ Look at Fig 5 of Shapiro + Fedorovich (2008).

Answer to above question: no. Actually the inexorable upward growth of the B and V boundary layers is even more pronounced in the turbulent simulation.

More on the Coriolis problem:

So... what can give us well behaved (boundary-layer-like) B and V? What about making provision for a synoptic-scale pressure gradient force? Egger (1985) looked at that and it did give well-behaved B but only for unrealistically huge (> 100 m/s) geostrophic winds.

Still More on the Coriolis problem

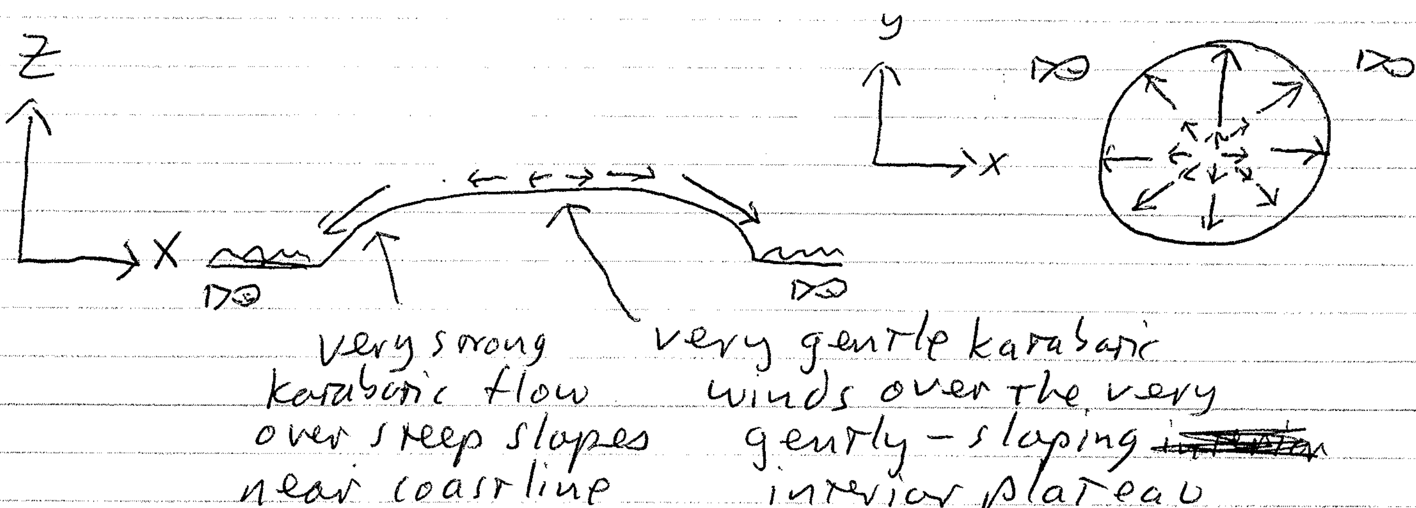
Egger (1985) also showed that a well-behaved soln for B, V could be obtained (without pgf) if one put in a linear radiative damping term in the thermodynamic energy equation

$$0 = U N^2 \sin\alpha + \kappa \frac{d^2 b}{dz^2} + \underbrace{\text{const}\,(b - b_{ref})}_{\substack{\text{linear damping} \\ b_{ref} = \text{reference value}}}$$

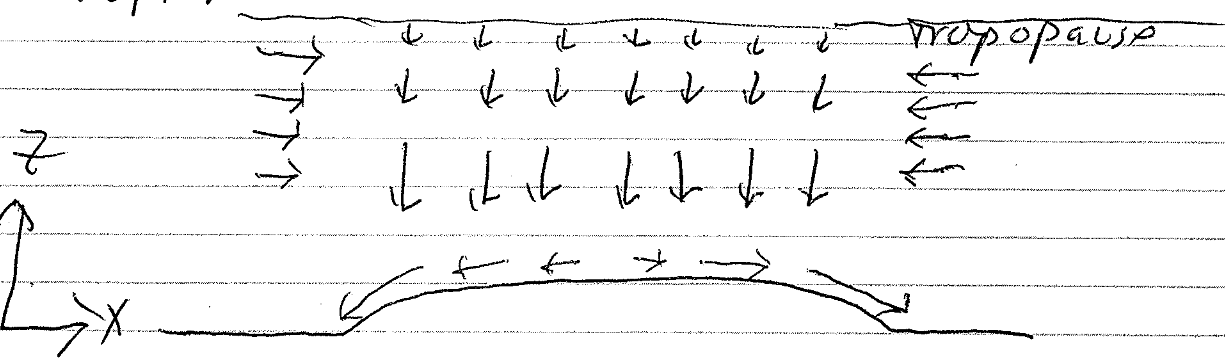
... it works but it's rather ad-hoc. A crude way of parameterizing radiation effects (or other effects not explicitly accounted for in the 1D framework).

In the real atmosphere the Coriolis force is profoundly important in the development of the polar vortex over Antarctica. Consider katabatic flow over the ice sheets that cover much of the Antarctic continent:

<u>vertical cross-section:</u> <u>plan view:</u>



Associated with this spatially-variable (divergent) katabatic flow is <u>entrainment</u> of air into the katabatic boundary layer and <u>subsidence aloft</u> and associated <u>convergence</u> aloft:



Associated with this wind field is <u>upper level stretching</u> $\left(\frac{\partial w}{\partial z} > 0\right)$. w is <u>negative</u> but it <u>increases</u> in value as z increases from mid-levels up to troposphere.

Parcels descend and stretch.

So in vertical vorticity eq^n: $\frac{\partial \zeta}{\partial t} = \ldots (\zeta + f) \frac{\partial w}{\partial z}$

the stretching term generates relative vertical vorticity ξ from the earth vorticity f.

[$f<0$ in S. hemisphere, $\frac{\partial w}{\partial z}>0$ aloft above Antarctica. So $f\frac{\partial w}{\partial z}<0$ $\therefore$ $\frac{\partial \xi}{\partial t}<0$ so negative ξ is generated. So ξ has same sense as f. So cyclonic relative vert vorticity has been generated. This leads to development of an upper level cyclonic polar vortex (see James (1989) for details).]

Final comment on Antarctic katabatic flows

Due to slight variations in topography, large expanses of gentle continental katabatic flow get funneled near the coast into valleys and gulleys. Get locally intense (>100 mph) long lasting wind storms lasting days or weeks. These winds continually break up sea ice and transport the ice down-wind. In coastal regions where katabatic winds are unusually intense and long-lived, sea is ice-free most of the year. More precisely, the sea ice is continually forming, being broken up and transported away. These zones of ~~ice sea~~ ice free water surrounded by ice are called polynyas.
As sea water freezes, salt is "rejected" and the density of the water increases. This heavy water sinks, and flows along the ocean floor in massive currents. This katabatic-induced production of Antarctic Bottom Water is an important feature of the global ocean circulation system.

→ See Figure 3. of Gordon (2001) article on Bottom Water Formation

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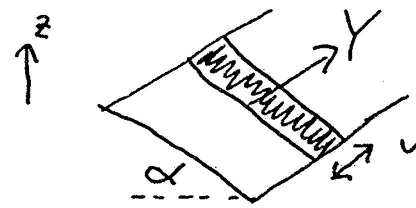
– 3 LLJ handouts + 1 katabatic paper (SBF)

S.+F. (2008) showed B and V could be well-behaved (boundary-layer like; decays to 0 as $Z \to \infty$) with Coriolis force included if edge effects are included. Provision made for surface cooling confined to a strip (thermal band) running down the slope:



← This flow is inhomogeneous in Y direction

See our paper for details.

Now lets look at inhomogeneous katabatic flows without Coriolis force. In general, these flows

can arise from spatially-inhomogeneous surface forcings in X and/or Y direction. Can arise from irregular ice/snow cover, snow/no-snow boundaries, topographic shading, variations in amount or type of vegetative covering, variations in land use, variations in soil moisture (e.g. due to gradient of surface rainfall), etc.

Consider thermo eq^n:

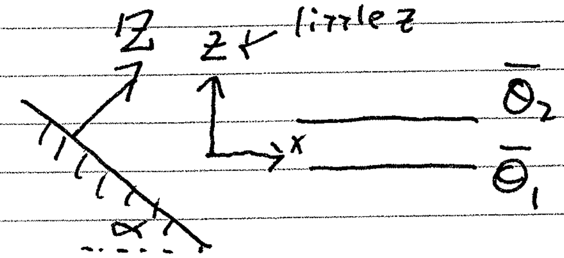
$$\frac{\partial b}{\partial t} + \vec{v} \cdot \nabla b = -\vec{v} \cdot \nabla \bar{\Theta} \frac{g}{\Theta_r} + K \nabla^2 b$$

where $\bar{\Theta} = \bar{\Theta}(z)$ is environmental pot. temp.

If there is no imposed synoptic-scale flow (so flow is driven entirely by surface forcings) and if $b \to 0$ as $Z \to \infty$ (as we expect, on physical grounds) then thermo eq^n reduces to:

$$\vec{v} \cdot \nabla \bar{\Theta} = 0 \quad (\text{as } Z \to \infty)$$

In the Prandtl scenario (homogeneous in X and Y dir^{ns}), $\vec{v} \to 0$ as $Z \to \infty$ and this allowed the above cond^n to be satisfied. But if buoyancy forcing is not homogeneous, there's no guarantee $\vec{v} \to 0$ as $Z \to \infty$. However, even if $\vec{v}$ doesn't go to 0, must have $\vec{v}\cdot\nabla\bar{\theta} = 0$ as $Z \to \infty$



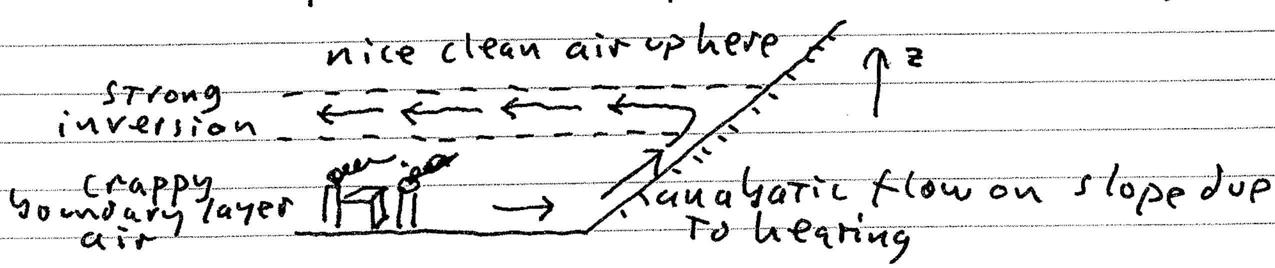
$\uparrow \nabla\bar{\theta}$ is in $\hat{k}$ dir^n (True vertical, not slope-normal)

$\therefore\ \vec{v}\cdot\hat{k} = 0$ as $Z \to \infty$

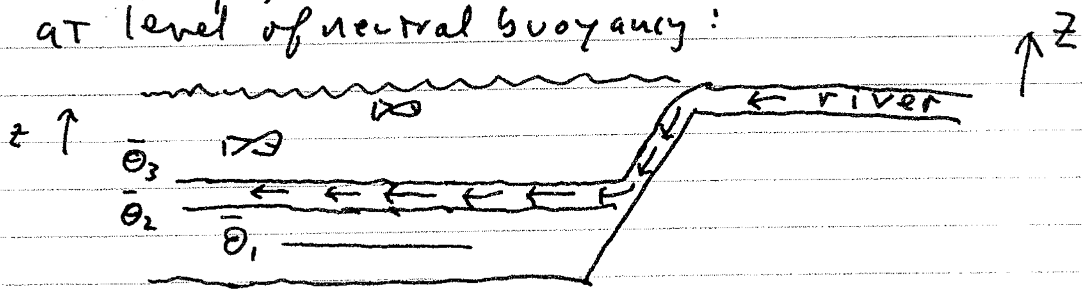
$\therefore\ w = 0$ as $Z \to \infty$. So vertical velocity component must vanish as $Z \to \infty$. But can have horizontal motion! i.e. flow along environmental isentropes.

The existence of fluid layers moving horizontally towards or away from sloping lateral boundaries has been observed in a variety of stratified fluid flows.

e.g. horizontal intrusion of boundary-layer air into environment at level of neutral buoyancy is important in air pollution meteorology:



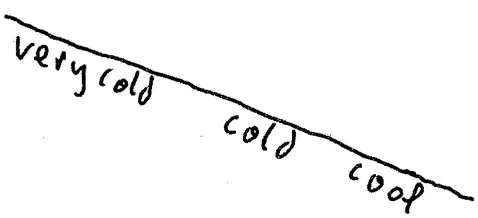
e.g. horizontal intrusion of cold dense river water into lakes/reservoir. The cold river water flows down slope, mixes w/ environment, then flows out laterally at level of neutral buoyancy:



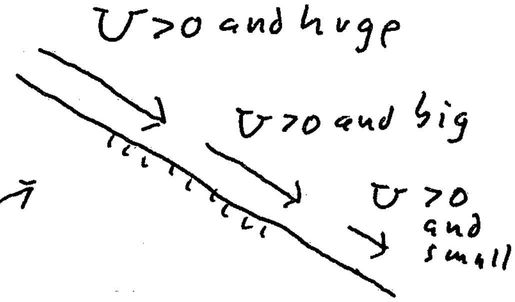
Katabatic flow along a differentially-cooled sloping surface

Shapiro + Fedorovich (2007) examined katabatic flow in which the surface buoyancy varied linearly with distance down the slope (X). Considered two cases:

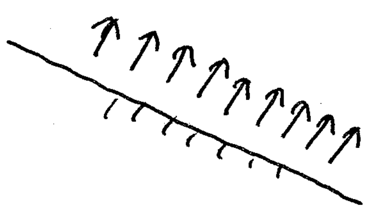
$\boxed{\text{case (i)}}$ surface buoyancy is negative but decreases in magnitude down the slope (i.e. buoyancy increases down the slope).



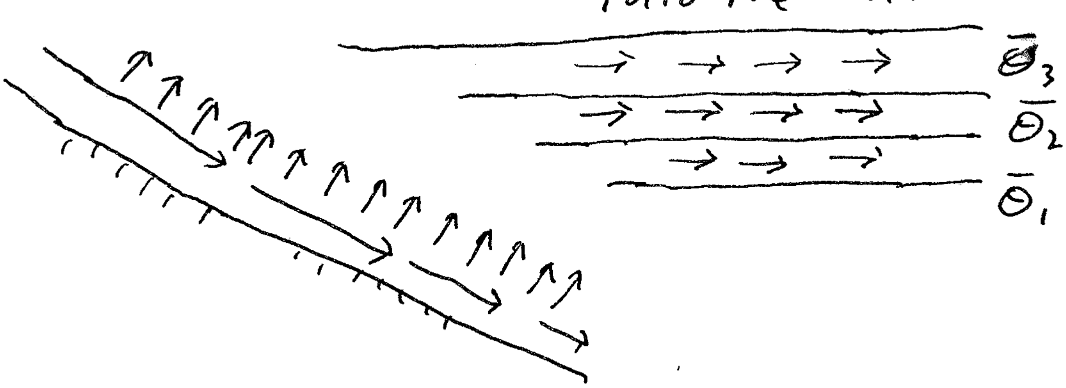
so near-sfc katabatic flow looks like



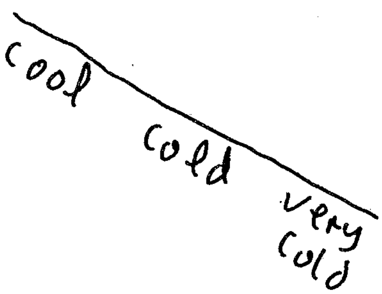
In this case $\frac{\partial U}{\partial X} < 0$. So from mass conservation $\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0$ and impermeability cond^n, ~~and that~~ we deduce that W must be positive at low levels above the slope:



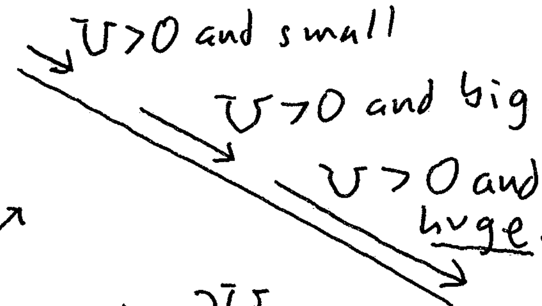
So we see that air must be leaving the katabatic boundary layer through it's top. Where does it go? Has to flow out horizontally into the environment:



$\boxed{\text{case (ii)}}$ surface buoyancy is <u>negative</u> and <u>increases</u> in magnitude down the slope (so buoyancy <u>decreases</u> down the slope).



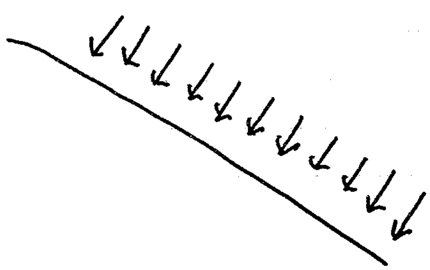
so near-surface karabatic flow looks like



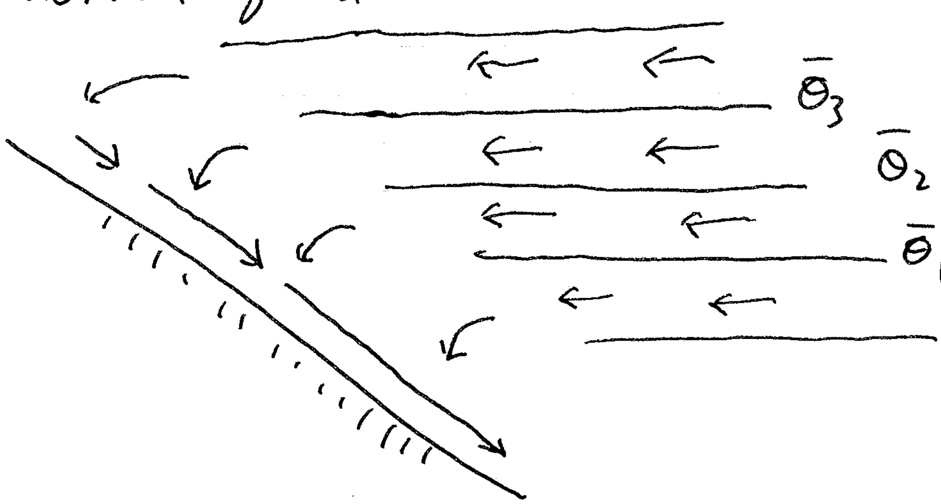
$$\therefore \frac{\partial U}{\partial X} > 0$$

$$\therefore \frac{\partial W}{\partial Z} < 0$$

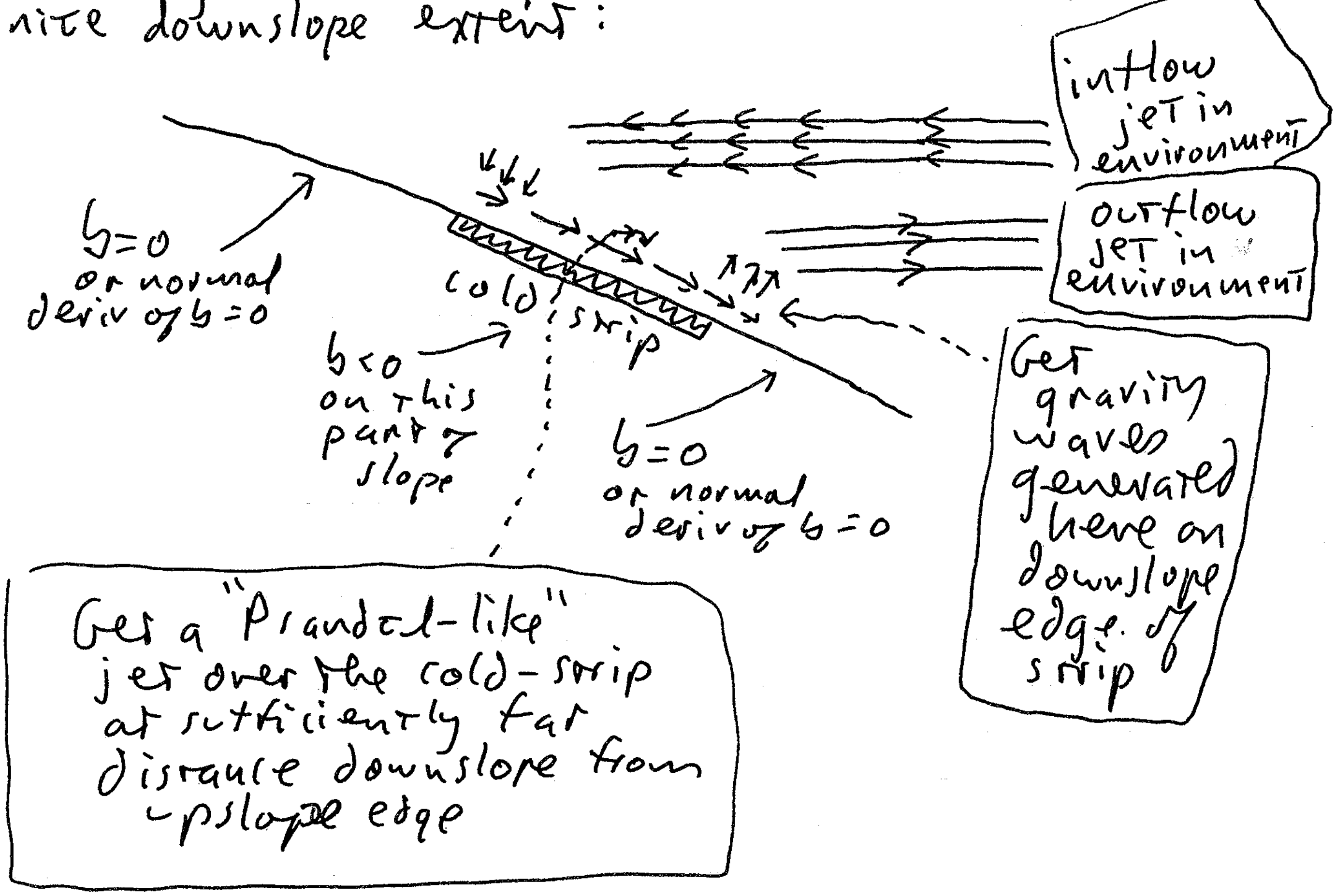
So, after int. w.r.t. Z and applying impermeability cond^n ($W(0)=0$) we see that W is <u>negative</u> at low levels above the slope:



In this case, air is getting sucked into the karabatic boundary layer from the environment. Get horizontal streaming motion of environmental air into b.l.:



Bryan Burkholder (SoM student) has studied flow induced by a cold strip (thermal band) of finite downslope extent:



Gravity waves appear to be generated continually near downslope edge of strip. These waves disturb flow throughout the whole domain.

So, very local surface conditions (inhomog forcing) can disturb the environment through gravity-wave production and production of the inflow and outflow ~~gabs~~ jets. These effects are felt very far from the cold strip.