METR 5123, Advanced Atmospheric Dynamics II
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REVIEW OF SURFACE GRAVITY WAVES

First, we'll review some notation and concepts that will be applicable to many kinds of waves (not just surface gravity waves).

Often we'll work w/ waves of the form:

$$\text{some flow property} \sim a \sin\left[\frac{2\pi}{\lambda}\left(x - ct\right)\right] \quad \text{or} \quad a \cos\left[\frac{2\pi}{\lambda}\left(x - ct\right)\right]$$

a is <u>amplitude</u>

$\frac{2\pi}{\lambda}\left(x - ct\right)$ is <u>phase</u> of wave

λ is <u>wavelength</u>

c is <u>phase speed</u>

Why consider waves of this form?

-- Many waves in atmosphere "look" like sines or cosines.

-- Many waves approximately satisfy <u>linear const coeff</u> <u>odes and pdes</u> that permit sin and cos solns. Can use <u>Fourier analysis</u> to get solns of complicated problems by summing sin and cos solns of various amplitudes and wavelengths.

A closer look at wave parameters:

phase: $\frac{2\pi}{\lambda}(x - ct)$

Wave repeats itself when phase changes by 2π. So, at a fixed moment in time, phase changes by 2π when x changes by λ. Hence the name wavelength for λ.

$k \equiv \frac{2\pi}{\lambda}$ is wavenumber, # waves in (dimensional) length of 2π.

e.g., if $\lambda = \pi$ (meters) then there are 2 waves in 2π meters.
$k = \frac{2\pi}{\pi\, m} = 2\ m^{-1}$.

Think: long waves --> small k
short waves --> big k

At a fixed point, phase changes by 2π when time changes by $\frac{\lambda}{c}$
So wave period is: $T \equiv \frac{\lambda}{c}$.

$\nu \equiv \frac{1}{T}$ is frequency, # of oscillations per unit time

$\omega \equiv \frac{2\pi}{T} = 2\pi\nu$ is circular (or radian) frequency.

Since $T = \frac{\lambda}{c} \rightarrow \omega = 2\pi \frac{c}{\lambda} = kc$

$\therefore \boxed{c = \frac{\omega}{k}}$

alternate expressions:

$$\sin\frac{2\pi}{\lambda}(x - ct)$$

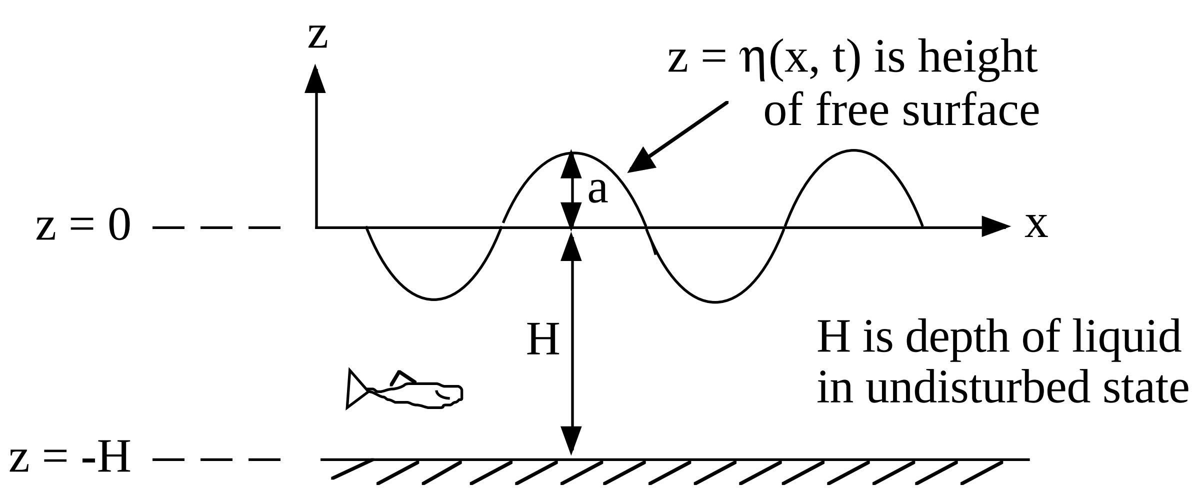
$$\sin[k(x - ct)]$$

$$\sin(kx - \omega t)$$

Motion of crests/troughts is motion of a geometric pattern. <u>Fluid does not generally move with the wave pattern</u>, e.g. Rossby waves, e.g., ocean waves might propagate toward shore at 20 m/s, but fluid moves toward and away from shore at ~ 1 m/s.

<u>Surface gravity waves</u>

Consider 2-D motion in the x-z plane of a liquid w/ a free surface (e.g. air/sea interface) [a pool of cold air underlying warm air is similar but more complicated]



-- assume disturbance is "infinitesimal", i.e., a is so small that:

$$a \ll H \quad (a/H \ll 1)$$

and $$a \ll \lambda \quad (a/\lambda \ll 1)$$

Because of this, nonlinear accel^n terms are very small compared to local derivs (products of small quantities are really small)

-- neglect friction

-- assume T << rotation period of earth, or equivalently $\omega \gg f$
∴ can safely neglect Coriolis force.

-- assume fluid was initially at rest (so irrotational) and waves created irrotationally. So from Kelvin's Th^m, the flow will always be irrot. So $\vec{\omega}(t) = 0$

$$\therefore \quad \boxed{\vec{u}(t) = \nabla\phi(t)}$$

-- assume flow is incompressible, $\nabla \cdot \vec{u} = 0$

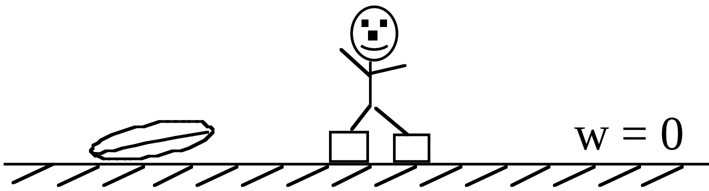
Substituting in $\vec{u} = \nabla\phi$ we get:

$$\nabla \cdot \nabla\phi = 0$$

∴ $\nabla^2\phi = 0$ Laplace's eqn.

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0}$$

Need b.c.s to solve it.
At bottom: impermeability cond^n:



w = 0 at z = - H

$$\therefore \quad \boxed{\frac{\partial \phi}{\partial z} = 0 \text{ at } z = -H}$$

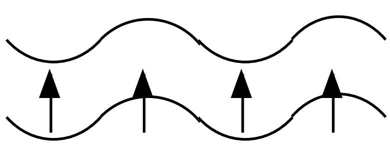
On top (free surface): a <u>kinematic</u> b.c. and a <u>dynamic</u> b.c.

<u>Top Kinematic b.c</u>.: A fluid element on free sfc remains on that sfc no matter how sfc moves or deforms. $\therefore$ $z_{\text{parcel on sfc}} = \eta$

$$\therefore \quad w_{\text{parcel on sfc}} = \frac{D}{Dt} z_{\text{parcel on sfc}} = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}$$

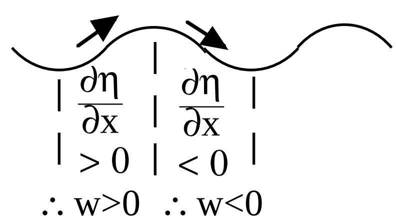
$$\therefore \quad \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \quad \text{on } z = \eta(x,t)$$

sfc moving upward:



$\partial\eta/\partial t > 0$ $\therefore$ $w > 0$

Stationary sfc with u>0

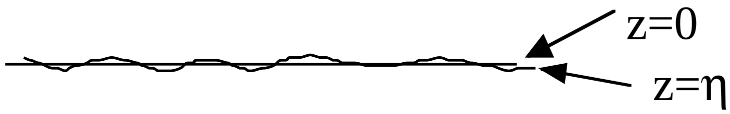


For infinitesimal waves, neglect products of small quantities (nonlinear terms). Neglect $u\,\partial\eta/\partial x$.

$$\therefore \quad \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{on } z = \eta(x,t) \quad [\text{still nonlinear: } \eta \text{ is affected by flow}]$$

Exapnd l.h.s. in a Taylor series about z = 0:

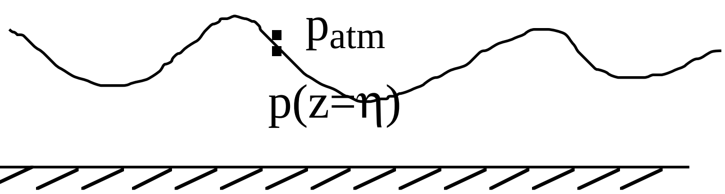
$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \underbrace{\left.\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)\right|_{z=0} \eta + \ldots}_{\text{neglect these terms since } \eta \text{ is small}}$$



$$\therefore \quad \boxed{\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \text{ on } z = 0} \quad \underline{\text{Linearized}} \text{ kinematic b.c. on free sfc.}$$

[Neglected a nonlinear term <u>and</u> put b.c. at z=0 instead of z=η]

<u>Dynamic b.c.</u> on free surface: pressure is <u>continuous</u> across air/sea interface. So pressure on liquid side of interface = atm pressure.



$$p(z = \eta(x,t)) = p_{atm}$$

Translate this b.c. into a b.c. on ϕ, using Bernoulli's eq^n for unsteady, irrot flow:

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \frac{p}{\rho} + gz = C \quad \text{(same const everywhere)}$$

Apply it on free sfc z=η (liquid side), w/ p=p_{atm}, and neglect q^2 :

$$\therefore \quad \frac{\partial \phi}{\partial t} + \frac{p_{atm}}{\rho} + g\eta = C$$

$$\therefore \quad \frac{\partial \phi}{\partial t} + g\eta = \left(C - \frac{p_{atm}}{\rho}\right) = \text{const}$$

Set const = 0. [Flow doesn't care about it. Equivalently, define $\phi_{new} = \phi_{old} + \text{const}\, t$, and plug into above eq^n. The const cancels out, but since $\vec{u} = \nabla\phi$ and $\nabla\phi_{new} = \nabla\phi_{old}$, the flow is unchanged.]

$$\therefore \quad \boxed{\frac{\partial \phi}{\partial t}} + g\eta = 0 \quad \text{at } z = \eta$$

↓

Expand it in a Taylor series in z about z = 0:

[neglect since η is small]

$$\therefore \quad \left.\frac{\partial \phi}{\partial t}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial t}\right|_{z=0} + \left.\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)\right|_{z=0} \eta + \text{h.o.t.}$$

∴ <u>linearized</u> <u>dynamic</u> free sfc b.c. is:

$$\boxed{\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } z = 0}$$

Want to solve $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0$ subject to the above b.c.

Consider a "wavy" pattern for η: $\eta = a\cos(kx - \omega t)$

From either top b.c., suspect $\phi \propto \sin(kx - \omega t)$ w/ a z dependence.

Trial solution: $\phi = f(z)\sin(kx - \omega t)$ [will it work?]

Plug it into Laplace's eq^n, get:

$$-k^2 f \sin(kx-\omega t) + \frac{d^2 f}{dz^2}\sin(kx-\omega t) = 0$$

$\therefore \quad \dfrac{d^2 f}{dz^2} - k^2 f = 0$ a 2nd order linear const coeff ode

Trial soln for f: $f = e^{mz}$. Plug into ode, get:

$$m^2 - k^2 = 0\,, \quad \therefore \quad m = k \text{ or } -k$$

So general soln for f is: $f = A\,e^{kz} + B\,e^{-kz}$

and so ϕ becomes:

$$(*) \quad \phi = \left(A\,e^{kz} + B\,e^{-kz}\right)\sin(kx-\omega t)$$

Now use b.c. to pin down A, B.

Lower kinematic b.c. is

$$\frac{\partial\phi}{\partial z} = 0 \text{ at } z = -H\,.$$

Apply it in (*), get

$$\left(A\,k\,e^{kz} - k\,B\,e^{-kz}\right)\Big|_{z=-H} \sin(kx-\omega t) = 0$$

$$\therefore \quad A\,k\,e^{-kH} - k\,B\,e^{kH} = 0$$

$$\therefore \quad \boxed{A = B\,e^{2kH}}$$

Top linearized kinematic b.c. is

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} \text{ on } z = 0 \,.$$

Apply it in (*), get

$$\therefore \left(Ake^{kz} - kBe^{-kz}\right)\Big|_{z=0} \sin(kx - \omega t) = -a(-\omega)\sin(kx - \omega t)$$

$$\therefore \boxed{k\left(A - B\right) = a\omega}$$

2 linear algebraic eq^{ns} for 2 unknowns, A, B. Solution is:

$$A = \frac{a\omega}{k\left(e^{2kH} - 1\right)} e^{2kH}, \qquad B = \frac{a\omega}{k\left(e^{2kH} - 1\right)}$$

$$\therefore \quad \phi = \frac{a\omega}{k\left(e^{2kH} - 1\right)}\left(e^{2kH} e^{kz} + e^{-kz}\right)\sin(kx - \omega t)$$

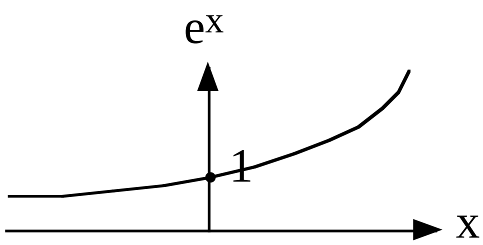
factor out e^{kH} from top and bottom, and mult by $\frac{2}{2}$

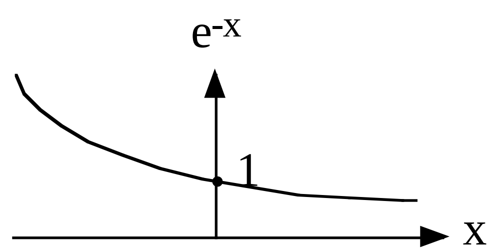
$$\therefore \quad \phi = \frac{a\omega}{k}\frac{2}{e^{kH} - e^{-kH}}\left(\frac{e^{k(z+H)} + e^{-k(z+H)}}{2}\right)\sin(kx - \omega t)$$

$$\therefore \quad \boxed{\phi = \frac{a\omega}{k} \frac{\cosh[k(z+H)]}{\sinh(kH)} \sin(kx - \omega t)}$$

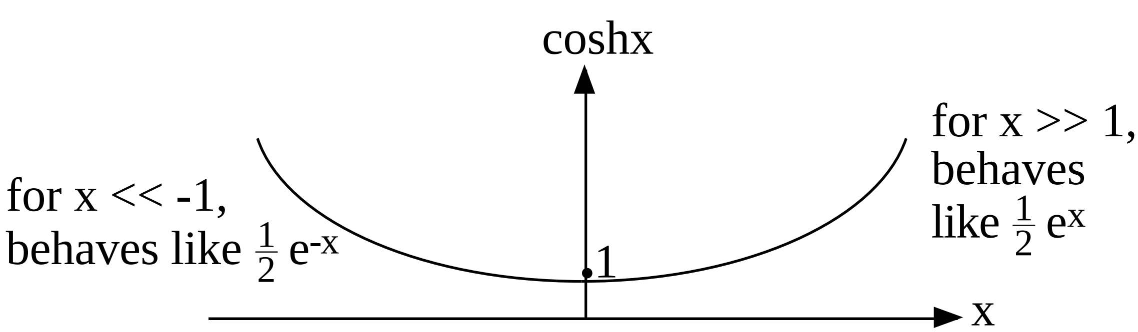
----- scratch paper, review of hyperbolic functions

here's how e^x and e^{-x} behave:

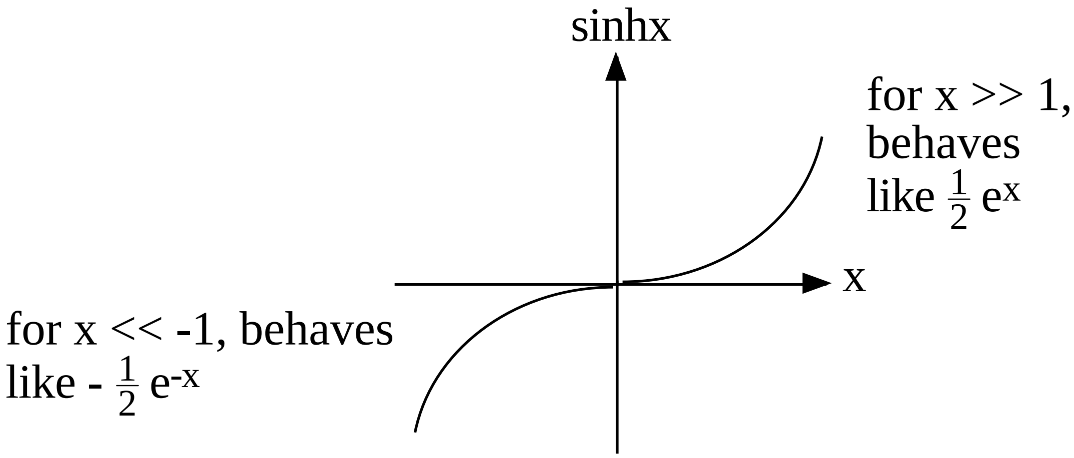




Now define $\boxed{\cosh x \equiv \frac{e^x + e^{-x}}{2}}$, the <u>average</u> of e^x and e^{-x}:



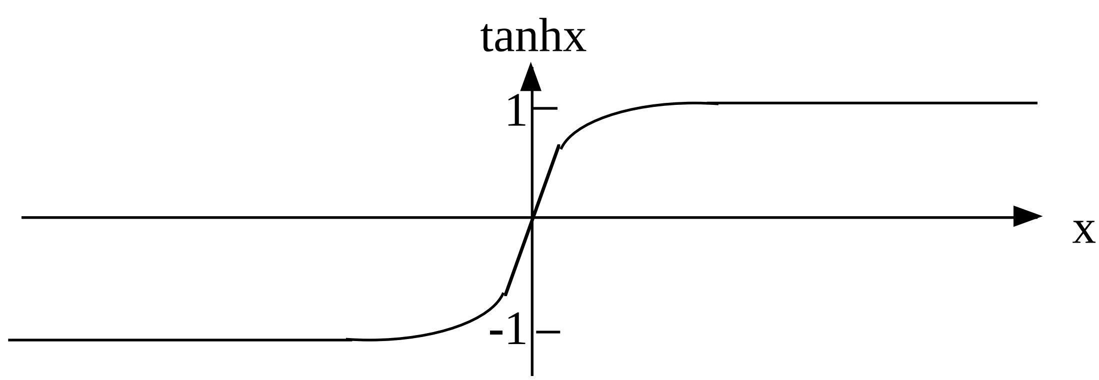
Now define $\boxed{\sinh x \equiv \frac{e^x - e^{-x}}{2}}$, the <u>difference</u> btw e^x and e^{-x} (divided by 2):



$$\frac{d}{dx}\cosh x = \frac{1}{2}\frac{d}{dx}\left(e^{x}+e^{-x}\right) = \frac{1}{2}\left(e^{x}-e^{-x}\right) = \sinh x$$

$$\frac{d}{dx}\sinh x = \frac{1}{2}\frac{d}{dx}\left(e^{x}-e^{-x}\right) = \frac{1}{2}\left(e^{x}+e^{-x}\right) = \cosh x$$

Now define $\boxed{\tanh x \equiv \frac{\sinh x}{\cosh x}}$



------------- end of scratch paper

$$u = \frac{\partial\phi}{\partial x} = a\omega\frac{\cosh[k(z+H)]}{\sinh(kH)}\cos(kx-\omega t)$$

$$w = \frac{\partial\phi}{\partial z} = a\omega \frac{\sinh[k(z+H)]}{\sinh(kH)} \sin(kx - \omega t)$$

Get p from linearized Bernoulli eq^n:

$$\frac{p}{\rho} = \text{const} - gz - \frac{\partial\phi}{\partial t}$$

$$= \text{const} - gz + \frac{a\omega^2}{k} \frac{\cosh[k(z+H)]}{\sinh(kH)} \cos(kx - \omega t)$$

Now apply linearized dynamic b.c.: $\frac{\partial\phi}{\partial t} + g\eta = 0$ on $z = 0$

$$\therefore \quad -\frac{a\omega^2}{k} \frac{\cosh(kH)}{\sinh(kH)} \cos(kx - \omega t) + g\, a \cos(kx - \omega t) = 0$$

$$\omega^2 = gk \tanh(kH)$$

$$\therefore \quad \boxed{\omega = \sqrt{gk \tanh(kH)}}$$ <u>dispersion relation</u> [$\omega = \omega(k)$]

$$c = \frac{\omega}{k}, \qquad \therefore\ c = \sqrt{\frac{g}{k} \tanh(kH)}$$

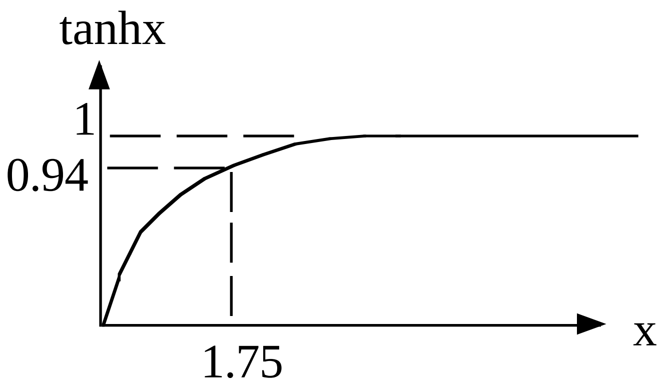
Note: c, ω indep of amplitude a (a feature of linear waves).

Now consider the limiting cases of "deep-water" and "shallow-water" surface gravity waves.

"deep-water" condition: $\boxed{\lambda \ll H}$ or $\frac{H}{\lambda} \gg 1$ or $kH \gg 1$

$$\therefore\ \tanh(kH) \approx 1$$

Actually, $\tanh(kH) \approx 1$ even for H/λ not too big:



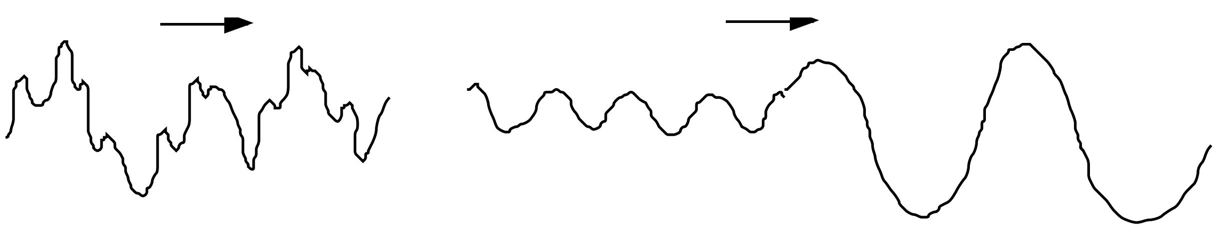
So, in practice, have "deep water" for $kH > 1.75$

So for deep water: $\omega_{deep} = \sqrt{gk}$, $\quad c_{deep} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$

$\therefore$ Longer waves have faster phase speeds. [No H dependence]

Consider a mixture of short <u>and</u> long waves in deep water.

After a while, waves sort themselves out (disperse):



superposition of short + long waves

short waves are slow pokes

long waves in front

When there's a storm way out at sea, the first waves to reach the shore are the long waves (low frequency waves).

"shallow-water" condition: $\boxed{\lambda >> H}$ or $\frac{H}{\lambda} << 1$ or $kH << 1$

$$\therefore \tanh(kH) = \frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}} = \frac{(1 + kH + ...) - (1 - kH + ...)}{(1 + kH + ...) + (1 - kH + ...)}$$

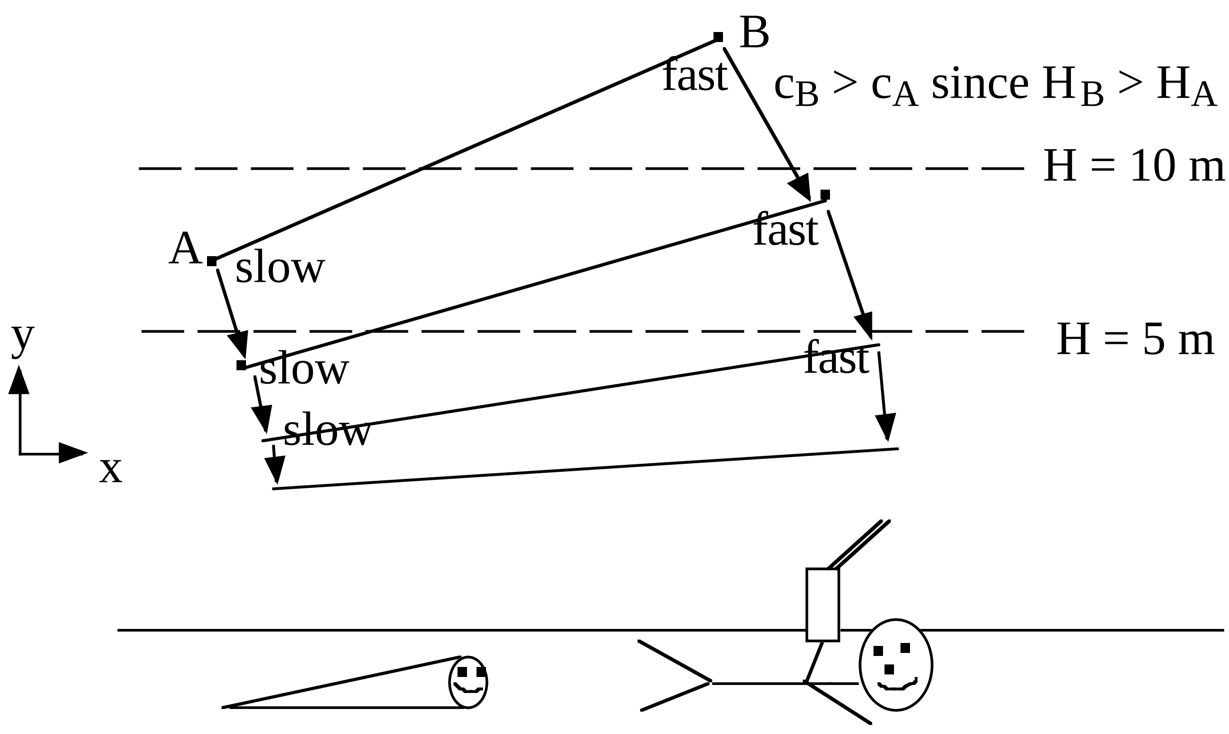
$$= \frac{2kH + \dots}{2 + \dots} \approx kH$$

$$\therefore \ \omega_{shallow} = \sqrt{gk\,kH} = k\sqrt{gH}$$

$$\therefore \ c_{shallow} = \frac{\omega_{shallow}}{k} = \sqrt{gH} \qquad \text{indep of k!}$$

So shallow-water waves are <u>non-dispersive</u>.
For shallow water waves, $c \downarrow$ as $H \downarrow$

Consider wave crests approaching a beach obliquely:



Wave crests turn as they approach beach, end up || to beach. A case of <u>wave refraction</u> (bending of wave fronts in inhomogeneous media -- in this case variable H(x)).

recall that for general surface gravity waves:

$$\omega = \sqrt{gk\tanh(kH)}\,, \qquad c = \sqrt{\frac{g}{k}\tanh(kH)}$$

and that for "shallow water": $kH \ll 1$ so we're led to

$$\omega_{shallow} = k\sqrt{gH}\,, \qquad c_{shallow} = \sqrt{gH}$$

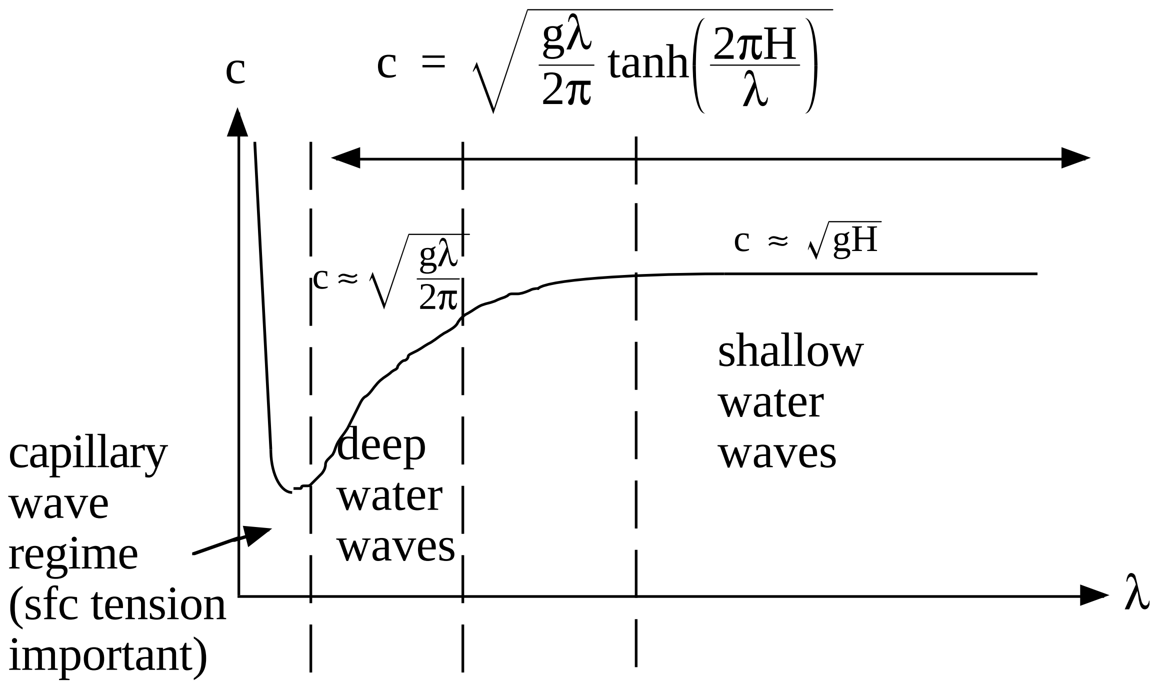
Now consider pressure in shallow water conditions:

$$\frac{p_{shallow}}{\rho} = \text{const} - gz + \frac{a\boxed{\omega^2}^{\approx k^2 gH}}{k}\frac{\boxed{\cosh[k(z+H)]}^{\approx 1}}{\boxed{\sinh(kH)}_{\approx kH}}\cos(kx - \omega t)$$

$$= \text{const} - gz + ag\cos(kx - \omega t)$$

$$= \text{const} - g(z - \eta)$$ <u>Hydrostatic pressure distribution.</u>

Now look at phase speed c for the general surface wave case (deep/shallow/whatever):



"deep" or "shallow" depends on λ relative to H. Water that's 100 m deep is "deep" for λ = 10m but "shallow" for λ = 1000 m.

Derive <u>streamfunction</u> for surface gravity wave [recall this is a 2D incomp flow]

$$\frac{\partial\psi}{\partial z} = u = \frac{\partial\phi}{\partial x} = a\omega \frac{\cosh[k(z+H)]}{\sinh(kH)} \cos(kx - \omega t)$$

integrate w.r.t. z:

$$(1) \quad \psi = \frac{a\omega}{k} \frac{\sinh[k(z+H)]}{\sinh(kH)} \cos(kx - \omega t) + F(x,t)$$

Similarly, $\frac{\partial\psi}{\partial x} = -w = \ldots$ Integrate w.r.t. x, to get:

$$(2) \quad \psi = \frac{a\omega}{k} \frac{\sinh[k(z+H)]}{\sinh(kH)} \cos(kx - \omega t) + G(z,t)$$

$F(x,t) = G(z,t)$ but f^n of x can't be a f^n of z -- so no x or z dependence. So $F(x,t) = G(z,t) = E(t)$. But E(t) is irrelevant since u, w only care about spatial derivs of ψ. So take $E(t) = 0$.

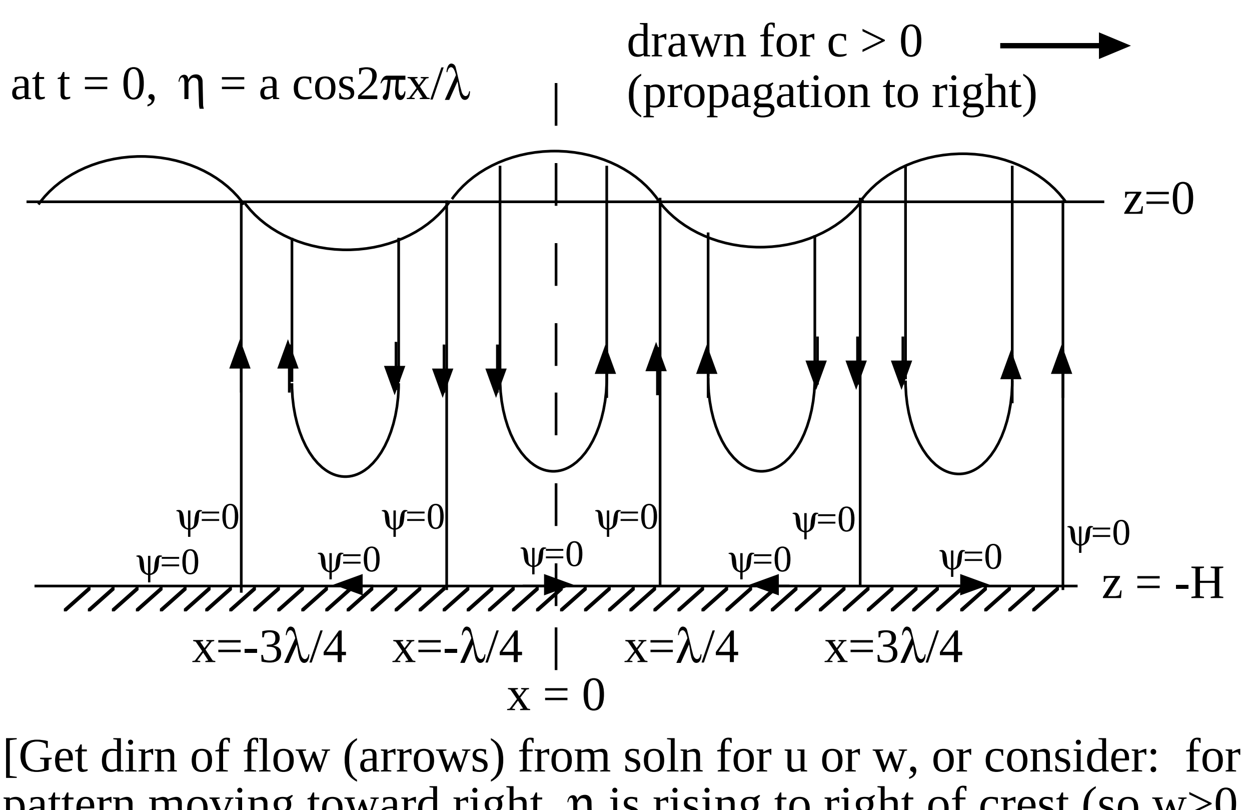
$$\therefore \quad \boxed{\psi = \frac{a\omega}{k}\frac{\sinh[k(z+H)]}{\sinh(kH)}\cos(kx - \omega t)}$$

Graph streamlines (ψ = const) at t = 0.

$\psi = 0$ for: $z = -H$ and for: $\boxed{k}\, x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$k \downarrow 2\pi/\lambda$

$$x = \pm\frac{\lambda}{4}, \pm\frac{3\lambda}{4}, \pm\frac{5\lambda}{4}, \ldots$$



[Get dirn of flow (arrows) from soln for u or w, or consider: for pattern moving toward right, η is rising to right of crest (so w>0 there) and η is falling to left of crest (so w<0 there). Arrows on bottom give sense of horiz conv/div needed to support this w field. Clearly parcel velocity differs from phase speed.].

Group Velocity

Concept of group velocity is appropriate for many different types of waves (not just surface gravity waves).

Consider 2 waves of equal amplitude and slightly different frequency and wavelength moving in same direction:

$$\omega_1 = \omega + \Delta\omega, \qquad k_1 = k + \Delta k$$

$$\omega_2 = \omega - \Delta\omega, \qquad k_2 = k - \Delta k$$

assume $\dfrac{\Delta\omega}{\omega} << 1, \qquad \dfrac{\Delta k}{k} << 1$

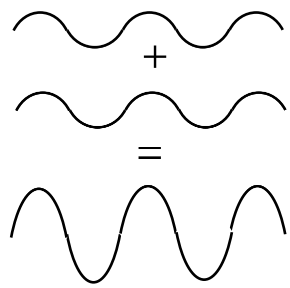
Because of dispersion relation, $\Delta\omega$ is related to Δk.

Mean frequency is: $\dfrac{\omega_1 + \omega_2}{2} = \dfrac{\omega + \Delta\omega + \omega - \Delta\omega}{2} = \omega$

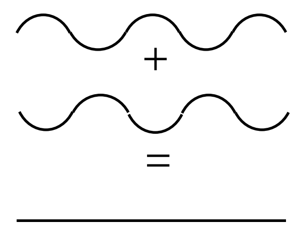
Mean wavenumber is: $\dfrac{k_1 + k_2}{2} = \ldots = k$

Where the waves are in phase (or nearly so) they combine to form a wave of twice amplitude. Where they're out of phase, they kill each other off.

2 waves in phase:



2 waves out of phase:



$$\eta = a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t)$$

$$= a\cos(kx - \omega t + \Delta k x - \Delta\omega t) + a\cos(kx - \omega t - (\Delta k x - \Delta\omega t))$$

$$= a\cos(kx - \omega t)\cos(\Delta k x - \Delta\omega t) \boxed{- a\sin(kx - \omega t)\sin(\Delta k x - \Delta\omega t)}$$

cancellation

$$+ a\cos(kx - \omega t)\cos(\Delta k x - \Delta\omega t) \boxed{+ a\sin(kx - \omega t)\sin(\Delta k x - \Delta\omega t)}$$

$$= 2a\cos(kx - \omega t)\cos(\Delta k x - \Delta\omega t)$$

$$\therefore \ \eta = \underset{\downarrow}{A} \ \underset{\downarrow}{\boxed{\cos(kx - \omega t)}} \qquad \text{where } A \equiv 2a\cos(\Delta k x - \Delta\omega t)$$

effective amplitude — carrier wave (mean wave)

<u>Effective amplitude A</u> is itself a wave with wavelength

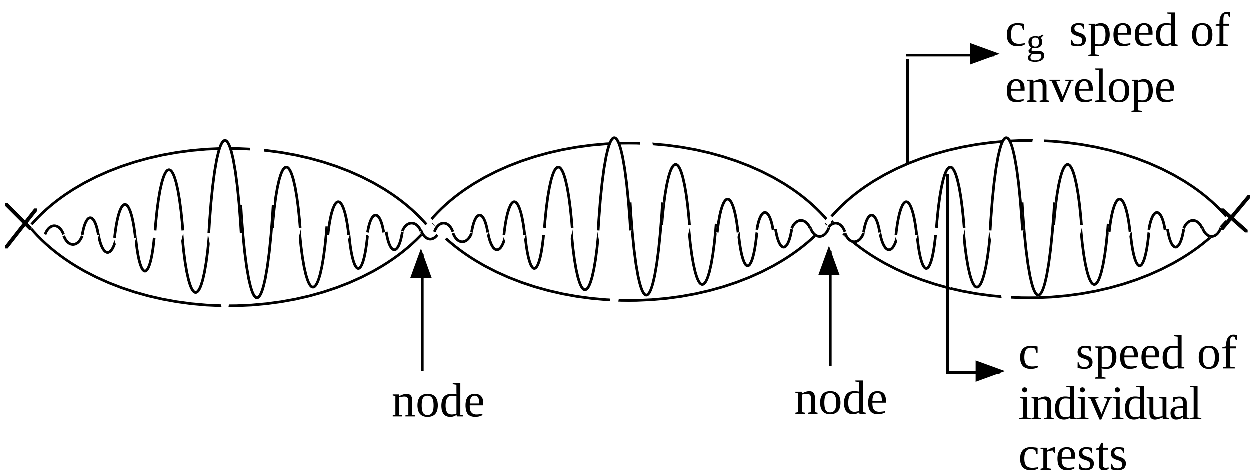
$$\lambda_{\text{amplitude}} = \frac{2\pi}{\Delta k} >> \frac{2\pi}{k} = \lambda_{\text{carrier wave}}$$

A propagates at speed $\dfrac{\Delta\omega}{\Delta k}$ where $\Delta\omega$ is related to Δk by dispersion rel^n. For small Δk, $\dfrac{\Delta\omega}{\Delta k} \rightarrow \dfrac{d\omega}{dk}$. Define $c_g \equiv \dfrac{d\omega}{dk}$

or $\vec{c}_g \equiv \frac{d\omega}{dk}\,\hat{i}$ Group velocity. A vector.

c is phase speed of crests [not a vector, see fig. 7.3 Kundu)

c_g is speed of envelope of crests.



Energy is trapped between nodes ∴ energy propagates at speed of nodes (speed of envelope), i.e. speed c_g, not phase speed c.

For deep-water sfc waves:

$$\omega = \sqrt{gk}$$

$$c = \sqrt{\frac{g}{k}}$$

$$c_g = \frac{d\omega}{dk} = \frac{1}{2\sqrt{k}}\sqrt{g}$$

$\therefore \quad \boxed{c_g = \frac{1}{2} c}$ [since $c > c_g$, individual crests move through envelope, die at nodes]

For <u>shallow water sfc waves</u>:

$$\omega = k\sqrt{gH}$$

$$c = \sqrt{gH}$$

$$c_g = \frac{d\omega}{dk} = \sqrt{gH}$$

$\therefore \quad \boxed{c_g = c}$