Trajectories associated with surface gravity waves

Trajectories are sol^{ns} of:

$$\frac{dx}{dt} = u(x,z,t), \quad \frac{dz}{dt} = w(x,z,t)$$

Plug in the surface wave sol^{ns} for u and w:

$$\frac{dx}{dt} = a\omega \frac{\cosh[k(z+H)]}{\sinh kH} \cos(kx-\omega t)$$

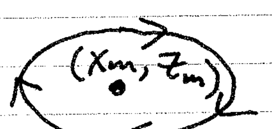
$$\frac{dz}{dt} = a\omega \frac{\sinh[k(z+H)]}{\sinh kH} \sin(kx-\omega t)$$

These are 2 first order coupled nonlinear odes.

Lets get an approximate solution valid for small amplitude waves. Anticipate that parcels don't move very far from their mean positions, so that the velocity of a parcel at time t is very close to the velocity at any other nearby point at same time t (e.g. the mean parcel location).

[Note: following notation differs from that in Kundu].

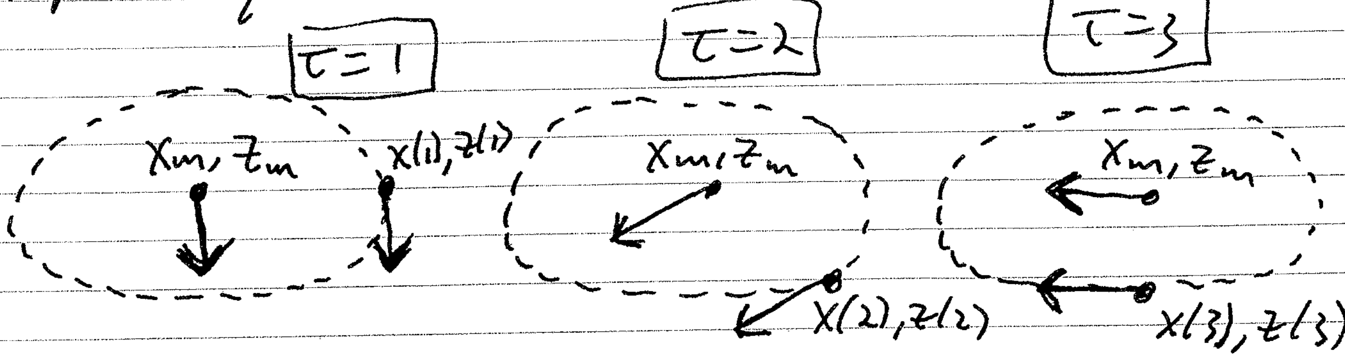
e.g. suppose this is the parcel trajectory:



where X_m, Z_m are the mean position coordinates of the trajectory of the parcel.

Then the velocity at the mean location X_m, Z_m at time t is roughly equal to the velocity of the parcel on its trajectory at the same time t.

Graphically:



In other words:

$$u(x(\tau), z(\tau), \tau) \approx u(X_m, Z_m, \tau)$$
$$w(x(\tau), z(\tau), \tau) \approx w(X_m, Z_m, \tau)$$

a linearizing approx.

So, on the r.h.s. of the traj. eq^{ns} approx: $x(\tau) \approx X_m$, $z(\tau) \approx Z_m$, get:

$$\frac{dx}{d\tau} = a\omega \frac{\cosh[k(Z_m+H)]}{\sinh kH} \cos(kX_m - \omega\tau)$$

$$\frac{dz}{d\tau} = a\omega \frac{\sinh[k(Z_m+H)]}{\sinh kH} \sin(kX_m - \omega\tau)$$

Now they're no longer non-linear. Just integrate wrt τ, get

$$(1) \quad x(\tau) = \text{const} - a \frac{\cosh[k(Z_m+H)]}{\sinh kH} \sin(kX_m - \omega\tau)$$

$$(2) \quad z(\tau) = \text{dif. const} + a \frac{\sinh[k(Z_m+H)]}{\sinh kH} \cos(kX_m - \omega\tau)$$

The constants of integration can be identified as the mean location X_m, Z_m of the parcel [average over a long time interval].

$\therefore$ const $= X_m$, dif const $= Z_m$

Want to take advantage of the fact that $\cos^2 A + \sin^2 A = 1$. So $\div$ (1) and (2) by appropriate factors, then square the resulting eq^{ns} then add 'em up:

$$\left(\frac{x(t) - X_m}{a\,\dfrac{\cosh[k(z_m+H)]}{\sinh kH}}\right)^2 + \left(\frac{z(t) - z_m}{a\,\dfrac{\sinh[k(z_m+H)]}{\sinh kH}}\right)^2 = 1$$

This is an ellipse.

At time τ when $x(\tau) = X_m$ then:

$$z(\tau) = z_m \pm a\,\frac{\sinh[k(z_m+H)]}{\sinh kH}$$

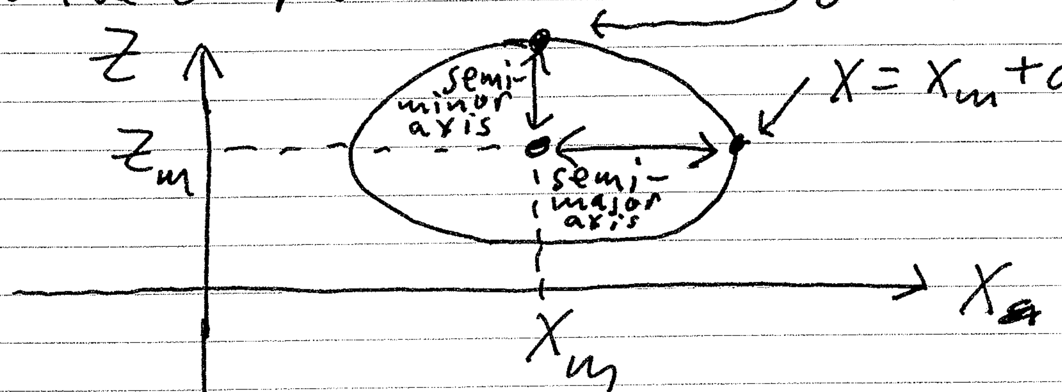
At time τ when $z(\tau) = z_m$ then:

$$x(\tau) = X_m \pm a\,\frac{\cosh[k(z_m+H)]}{\sinh kH}$$

So the ellipse looks like:

$$z = z_m + a\,\frac{\sinh[k(z_m+H)]}{\sinh kH}$$

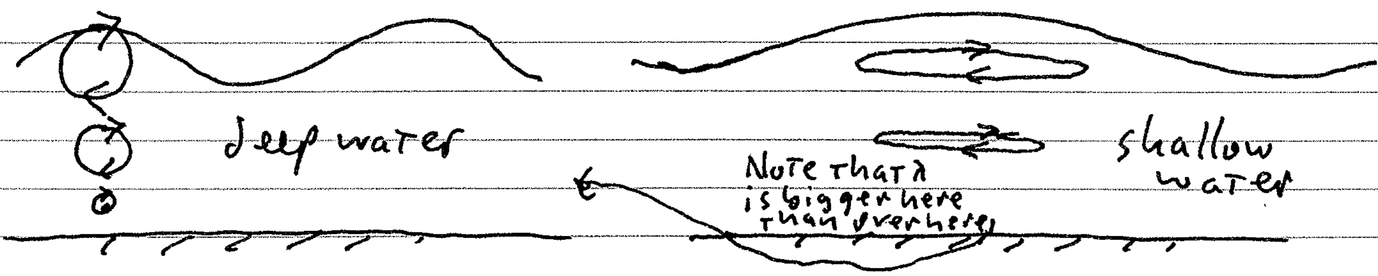
$$X = X_m + a\,\frac{\cosh[k(z_m+H)]}{\sinh kH}$$



In general, since $|\cosh[k(z_m+H)]| > |\sinh[k(z_m+H)]|$ the ellipse is elongated in the x-dir^n. [Think of the diagrams for cosh and sinh].

Note that both the semi-minor axis and semi-major axis decrease as $z\downarrow$. At the bottom ($z_m = -H$), the semi-minor axis is 0.

For deep water ($kH \gg 1$), the semi-major and semi-minor axes become similar so the trajectory is nearly a circle. But for shallow water ($kH \ll 1$), the semi-minor axis becomes much smaller than the semi-major axis.

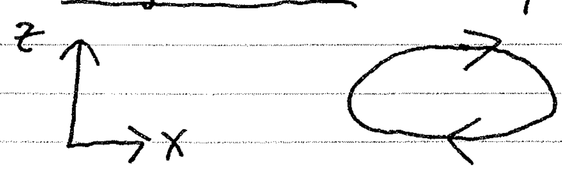


Stokes Drift

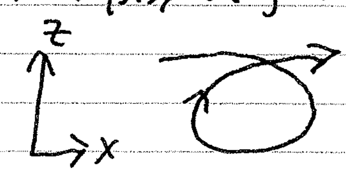
We'll consider it in a surface gravity wave but it's important in internal gravity waves as well. Occurs in the oceans, lakes and the atmosphere (usually in the mesosphere + stratosphere; important in case of vertically-propagating waves).

Idea: a parcel experiences a slow "drift" over time in the direction of the wave propagation even though the average velocity at a fixed point is 0.

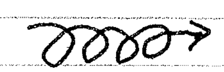
Using the linear sol^n for surface gravity waves followed by the further linearized trajectory analysis we get closed trajectories (ellipses)



But if we use the linear sol^n for surface gravity waves in a slightly more accurate (non-linear) trajectory analysis, we get:

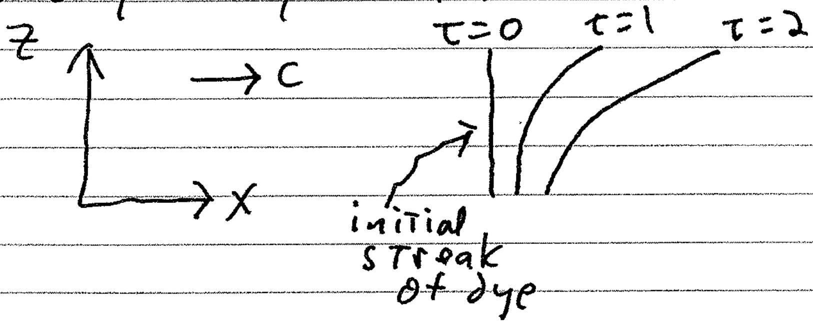


or:



So including nonlinearity in the trajectory analysis reveals that parcels "migrate" or "drift" with time. Get mass transport.

If you were to consider an initially vertical streak of dye (or "chum") it would shift laterally (and bend) due to Stokes drift (and fact that the amplitude of the oscillation decreases with increasing depth in the sfc gravity wave):



Stokes drift occurs because a parcel spends slightly more of its time moving forward (i.e. in the direction of the wave propagation) than moving backward.

Now lets calculate the drift speed.

Before, in the purely linear trajectory analysis we replaced the parcel velocity at time t by the velocity at a fixed point removed from the trajectory (the mean position of the trajectory, a fixed point that the parcel is never on). This is equivalent to treating u, w throughout the region of the trajectory as spatially constant (but varying in time as cos or sin).

Now we'll use a more accurate way of approximating the parcel velocity at time t. Essentially we'll treat u and w as varying linearly with x and z (and varying in time as cos or sin).

First note that since the parcel will be oscillating and drifting, the concept of a fixed point characterizing the mean location of the parcel is not applicable. So, if we do want to consider some reference location, lets work with something well-defined. How about: the initial (t=0) coordinates of the parcel, x_0, z_0.

Now need to be careful with notation (Lagrangian versus Eulerian).

Consider the parcel that started out ($\tau = 0$) at location x_0, z_0. At some arbitrary time τ the parcel is at the new location x, z. We can write the x-comp of the parcel velocity at time τ in Lagrangian form as:

$u_L(x_0, z_0, \tau)$

We can write the exact same thing in Eulerian notation as:

$u(x, z, \tau)$, with the understanding that these x, z are the coordinates of the parcel at time τ that used to be ($\tau = 0$) at x_0, z_0. So these x, z should be thought of as $x(x_0, z_0, \tau)$, $z(x_0, z_0, \tau)$.

So with this understanding of the notation, we can write:

$$(\star) \quad u_L(x_0, z_0, \tau) = u(x, z, \tau)$$

Now we're going to approx $u(x,z,\tau)$ by the form:

$$u(x,z,\tau) \approx u(x_0, z_0, \tau) + (x - x_0)\left.\frac{\partial u}{\partial x}\right|_{x_0,z_0} + (z - z_0)\left.\frac{\partial u}{\partial z}\right|_{x_0,z_0}$$

[Taylor expansion of Eulerian u in x and z about the point x_0, z_0 with higher order terms neglected].

This approximated form of u varies linearly in x and z.

Similarly, for the z-comp of this parcel's velocity at time τ, we can write:

$$(\star\star) \quad w_L(x_0, z_0, \tau) = w(x, z, \tau)$$

and we approximate $w(x,z,\tau)$ by:

$$w(x, z, \tau) \simeq w(x_0, z_0, \tau) + (x - x_0)\left.\frac{\partial w}{\partial x}\right|_{x_0, z_0} + (z - z_0)\left.\frac{\partial w}{\partial z}\right|_{x_0, z_0}$$

To get the components of the drift velocity we need to average $u_L(x_0, z_0, \tau)$ and $w_L(x_0, z_0, \tau)$

over time, say, over 1 wave period T (where $T = 2\pi/\omega$). Get same result if we average over arbitrary integer number of periods.

So, we want to estimate $\overline{u_L} \equiv \frac{1}{T}\int_0^T u_L(x_0, z_0 \tau)\, d\tau$

and $\overline{w_L} \equiv \frac{1}{T}\int_0^T w_L(x_0, z_0, \tau)\, d\tau$

Taking the time ave of $(\star)$ and $(\star\star)$, and substituting in the approximated forms of $u(x,z,\tau)$ and $w(x,z,\tau)$ we get:

$$\overline{u_L} = \frac{1}{T}\int_0^T u(x_0, z_0, \tau)\, d\tau + \frac{1}{T}\int_0^T (x - x_0)\left.\frac{\partial u}{\partial x}\right|_{x_0, z_0} d\tau + \frac{1}{T}\int_0^T (z - z_0)\left.\frac{\partial u}{\partial z}\right|_{x_0, z_0} d\tau$$

☺

$$\overline{w_L} = \frac{1}{T}\int_0^T w(x_0, z_0, \tau)\, d\tau + \frac{1}{T}\int_0^T (x - x_0)\left.\frac{\partial w}{\partial x}\right|_{x_0, z_0} d\tau + \frac{1}{T}\int_0^T (z - z_0)\left.\frac{\partial w}{\partial z}\right|_{x_0, z_0} d\tau$$

☺

Lets evaluate the sol^n for $\bar{u}_L$, $\bar{w}_L$ for the special case of deep water [$kH \gg 1$]

recall:

$$\omega = \sqrt{gk}$$

$$\text{so} \quad u = a\omega \frac{\cosh[k(z+H)]}{\sinh kH} \cos(kx - \omega t)$$

$$\cosh[k(z+H)] \approx \frac{e^{k(z+H)}}{2} \qquad \sinh kH \approx \frac{e^{kH}}{2}$$

$$\therefore \quad \boxed{u \approx a\omega e^{kz} \cos(kx - \omega t)}$$

similarly: $\boxed{w \approx a\omega e^{kz} \sin(kx - \omega t)}$

Take spatial derivs of these expressions and then evaluate at x_0, z_0

So:
$$\left.\frac{\partial u}{\partial x}\right|_{x_0, z_0} = -a\omega k e^{kz_0} \sin(kx_0 - \omega t)$$

$$\left.\frac{\partial u}{\partial z}\right|_{x_0, z_0} = a\omega k e^{kz_0} \cos(kx_0 - \omega t)$$

$$\left.\frac{\partial w}{\partial x}\right|_{x_0, z_0} = a\omega k e^{kz_0} \cos(kx_0 - \omega t)$$

$$\left.\frac{\partial w}{\partial z}\right|_{x_0, z_0} = a\omega k e^{kz_0} \sin(kx_0 - \omega t)$$

[digression: note that $\partial u/\partial z$ and $\frac{\partial w}{\partial x}$ are the same, so the y-comp vorticity $\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ is 0! So, no y-comp vort, but there is pronounced curvature of the trajectory, nearly circles in this case]

To evaluate ☺, also need to calculate $x - x_0$ and $z - z_0$ keeping in mind the x, z are on the trajectory. Can get a good estimate of them from the pure linear trajectory calculation:

$$\frac{dx}{d\tau} = u(x, z, \tau) \approx u(x_0, z_0 \tau)$$

$$\frac{dz}{d\tau} = w(x, z, \tau) \approx w(x_0, z_0 \tau)$$

∴ For deep water case:

$$\frac{dx}{d\tau} \approx a\omega e^{kz_0} \cos(kx_0 - \omega\tau)$$

$$\frac{dz}{d\tau} \approx a\omega e^{kz_0} \sin(kx_0 - \omega\tau)$$

integrate from $\tau = 0$ to arbitrary time, get

$$x - x_0 = -ae^{kz_0}\left[\sin(kx_0 - \omega\tau) - \sin kx_0\right]$$

$$z - z_0 = ae^{kz_0}\left[\cos(kx_0 - \omega\tau) - \cos kx_0\right]$$

[Note: Kundu's formulas for $x - x_0$, $z - z_0$ in Stokes drift section are not quite right: they don't yield right behavior at $\tau = 0$]

So, now let's evaluate the integrals in ☺

$$\frac{1}{T}\int_0^T u(x_0, z_0, \tau)\, d\tau = \frac{1}{T}\int_0^T a\omega e^{kz_0} \cos(kx_0 - \omega\tau)\, d\tau$$

$$= -\frac{a\omega e^{kz_0}}{\omega T}\left[\sin(kx_0 - \omega\tau)\right]_0^T$$

$$= -\frac{ae^{kz_0}}{T}\left[\sin(kx_0 - \omega T) - \sin kx_0\right]$$

$$= -\frac{ae^{kz_0}}{T}\left[\sin kx_0 - \sin kx_0\right] = 0$$

note:
$\omega T = \frac{2\pi}{T} T = 2\pi$

So $\sin(kx_0 - \omega T) = \sin(kx_0 - 2\pi) = \sin kx_0$

Now look at:

$$\frac{1}{T}\int_0^T (x-x_0)\left.\frac{\partial u}{\partial x}\right|_{x_0,z_0} d\tau$$

$$= \frac{1}{T}\int_0^T -ae^{kz_0}\left[\sin(kx_0-\omega\tau)-\sin kx_0\right]\left[-a\omega k e^{kz_0}\sin(kx_0-\omega\tau)\right]d\tau$$

$$= \frac{a^2\omega k e^{2kz_0}}{T}\int_0^T \sin^2(kx_0-\omega\tau)\,d\tau$$

$$- \frac{a^2\omega k e^{2kz_0}}{T}\sin kx_0 \underbrace{\int_0^T \sin(kx_0-\omega\tau)\,d\tau}_{\to 0}$$

$$= \frac{a^2\omega k e^{2kz_0}}{T}\int_0^T \frac{1-\cos[2(kx_0-\omega\tau)]}{2}\,d\tau$$

$$= \frac{a^2\omega k e^{2kz_0}}{T}\left[\frac{1}{2}T+0\right]$$

$$= \frac{a^2\omega k}{2}e^{2kz_0}$$

scratch paper:

$\sin^2 a + \cos^2 a = 1$

$\cos 2a = \boxed{\cos^2 a} - \sin^2 a$

$= 1 - 2\sin^2 a$

$\therefore \sin^2 a = \dfrac{1-\cos 2a}{2}$

also, for below:

$\cos 2a = \cos^2 a - \boxed{\sin^2 a}$

$= 2\cos^2 a - 1$

$\therefore \cos^2 a = \dfrac{1+\cos 2a}{2}$

Now look at:

$$\frac{1}{T}\int_0^T (z-z_0)\left.\frac{\partial u}{\partial z}\right|_{x_0,z_0} d\tau$$

$$= \frac{1}{T}\int_0^T ae^{kz_0}\left[\cos(kx_0-\omega\tau)-\cos kx_0\right]\left[a\omega k e^{kz_0}\cos(kx_0-\omega\tau)\right]d\tau$$

$$= \frac{a^2\omega k e^{2kz_0}}{T}\int_0^T \cos^2(kx_0-\omega\tau)\,d\tau - \frac{a^2\omega k e^{2kz_0}}{T}\cos kx_0 \underbrace{\int_0^T \cos(kx_0-\omega\tau)\,d\tau}_{\to 0}$$

$$= \frac{a^2\omega k e^{2kz_0}}{T}\int_0^T \frac{1+\cos[2(kx_0-\omega\tau)]}{2}\,d\tau$$

$$= \frac{a^2\omega k e^{2kz_0}}{T}\left[\frac{1}{2}T+0\right] = \frac{a^2\omega k}{2}e^{2kz_0}$$