Putting it together in ☺, get $\bar{u}_L$:

$$\bar{u}_L = 0 + \frac{a^2 \omega k}{2} e^{2kz_0} + \frac{a^2 \omega k}{2} e^{2kz_0}$$

$$\therefore \quad \boxed{\bar{u}_L = a^2 \omega k e^{2kz_0}}$$

Similarly, can show that:

$$\boxed{\bar{w}_L = 0}$$

So the drift velocity (a.k.a. mass transport velocity or Lagrangian mean velocity) is in the direction of the wave propagation. Its magnitude is proportional to wave frequency and the square of the wave amplitude (a nonlinear effect).

Standing Waves

Consider 2 propagating waves of equal amplitude and wavelength propagating in opposite directions.

For surface gravity waves the free surface displacement can be described by:

$$\eta = a \cos(kx - \omega t) + a \cos(kx + \omega t)$$

[If $k > 0$ and $\omega > 0$ then first wave propagates in $+x$ dir^n, while second wave propagates in $-x$ dir^n. Why? Consider, for example, the first wave. As $t \uparrow$ then x must $\uparrow$ in order that the phase be const i.e. for us to follow a given phase, like a crest or trough.]

use: $\cos(b-c) = \cos b \cos c + \sin b \sin c$
$\cos(b+c) = \cos b \cos c - \sin b \sin c$

to rewrite η as:

cancellation!

$$\eta = a\left(\cos kx \cos\omega t \;\boxed{+\; \sin kx \sin\omega t}\right)$$
$$+\, a\left(\cos kx \cos\omega t \;\boxed{-\; \sin kx \sin\omega t}\right)$$

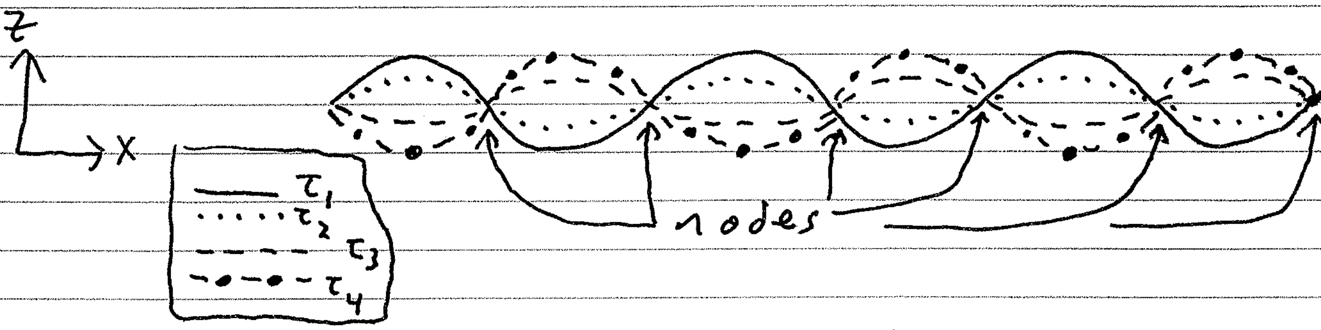
$\therefore$ $\boxed{\eta = 2a \cos kx \cos\omega t}$

"nodes" are the locations of zero surface displacement.

So, for this case we get nodes where $kx = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

The locations of these nodes does not change in time.

Look at sfc displacements at various times:



Wave looks like it's "breathing", not propagating.
But it's really 2 waves propagating in opposite directions.

This is called a standing wave (free sfc does not propagate).

We've already derived the relevant governing eq^n (Laplace's eq^n) and the linearized boundary cond^{ns} for this problem. Since the governing eq^{ns} (+b.c.) are linear and homogeneous, the sol^n for the standing wave is just the sum of the ~~two~~ sol^{ns} corresponding to the two individually propagating waves, Recall that the soln for u, w for one wave with free-surface displacement $\eta = a\cos(kx-\omega t)$ is:

$$\left.\begin{aligned} u &= a\omega \frac{\cosh[k(z+H)]}{\sinh kH}\cos(kx-\omega t) \\ w &= a\omega \frac{\sinh[k(z+H)]}{\sinh kH}\sin(kx-\omega t) \end{aligned}\right\} \text{ for } \eta = a\cos(kx-\omega t) \ \rightarrow$$

So, to get the soln for the wave moving in the opposite dirn, i.e. wave associated with surface displacement $\eta = a\cos(kx+\omega t)$, just change "$\omega$" to "$-\omega$" everywhere in the above soln, get:

$$\left.\begin{aligned} u &= -a\omega \frac{\cosh[k(z+H)]}{\sinh kH}\cos(kx+\omega t) \\ w &= -a\omega \frac{\sinh[k(z+H)]}{\sinh kH}\sin(kx+\omega t) \end{aligned}\right\} \text{ for } \eta = a\cos(kx+\omega t) \ \leftarrow$$

Now, to get the standing wave soln, add these two solns together.

$$u = a\omega \frac{\cosh[k(z+H)]}{\sinh kH}\underbrace{\left[\cos(kx-\omega t) - \cos(kx+\omega t)\right]}$$

$$\cos kx\cos\omega t + \sin kx \sin\omega t - (\cos kx\cos\omega t - \sin kx\sin\omega t)$$

$$= 2\sin kx \sin\omega t$$

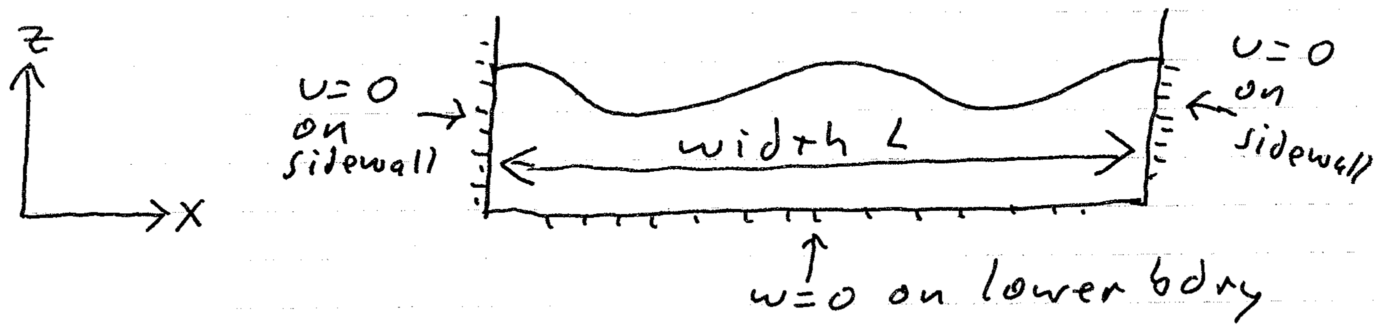
$$\therefore \boxed{u = 2a\omega \frac{\cosh[k(z+H)]}{\sinh kH}\sin kx\sin\omega t}$$

similarly: for w

$$\boxed{w = -2a\omega \frac{\sinh[k(z+H)]}{\sinh kH}\cos kx\sin\omega t}$$

Standing wave soln, i.e. for $\eta = 2a\cos kx\cos\omega t$

Note that $u=0$ for $\sin kx = 0$ regardless of time.
So ~~kx~~ $u=0$ for $kx = 0, \pi, 2\pi, \ldots$ Use that fact to construct standing wave sol^{ns} in a 2D (x,z) channel of width L:



Need to impose impermeability cond^n on lateral boundaries as well as on lower boundary (so $u=0$ on sidewalls, and $w=0$ on lower bdry).

In order to insure that $u=0$ on the left sidewall, choose coord system such that $x=0$ is location of that sidewall. Since $\sin kx = \sin k \cdot 0 = 0$ there, $u=0$ there.

With $x=0$ being left sidewall, the right sidewall must be at $x=L$. And since we want $u=0$ there, must have $\sin kL = 0$ $\therefore$ $kL = \pi, 2\pi, 3\pi \ldots$

$$\therefore \quad kL = (n+1)\pi \qquad n=0,1,2\ldots$$

but $k = \frac{2\pi}{\lambda}$ $\therefore$ $\frac{2\pi L}{\lambda} = (n+1)\pi$

$$\therefore \quad \boxed{\lambda = \frac{2L}{n+1}}$$

[error in formula for λ in Kundu]

So not just any old λ (or k) are allowed. Only special (discrete) wavelengths allowed:

$$\underset{(n=0)}{\lambda = 2L} \qquad \underset{(n=1)}{\lambda = L} \qquad \underset{(n=2)}{\lambda = \tfrac{2}{3}L} \qquad \text{etc.}$$

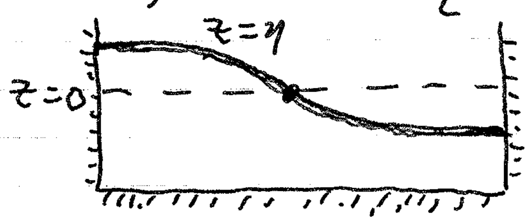
Dispersion relⁿ is still $\omega = \sqrt{gk \tanh kH}$ where $k = 2\pi/\lambda$ so corresponding to the set of discrete wavelengths is a set of discrete <u>natural frequencies</u>.

Look at the free-surface displacements for the different modes [a "mode" is a pattern of motion]:

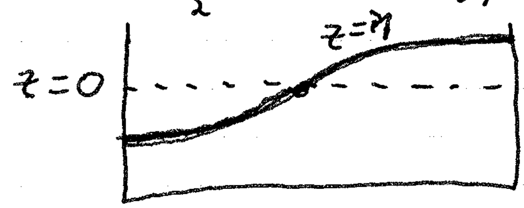
$\boxed{n=0 \text{ mode}}$ $\therefore \lambda = 2L, \quad k = \frac{2\pi}{\lambda} = \frac{\pi}{L}$

$\therefore \eta = 2a \cos\frac{\pi x}{L} \cos\omega t$

at $t=0$, $\eta = 2a\cos\frac{\pi x}{L}$:

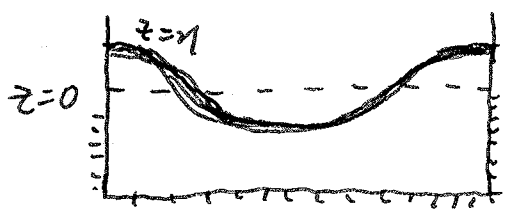


at $t = \frac{T}{2}$ (where $T = \frac{2\pi}{\omega}$): $\eta = -2a\cos\frac{\pi x}{L}$:

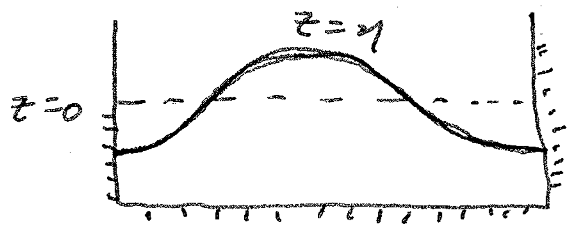


$\boxed{n=1 \text{ mode}}$ $\therefore \lambda = L, \quad k = \frac{2\pi}{\lambda} = \frac{2\pi}{L}$

at $t=0$, $\eta = 2a\cos\frac{2\pi x}{L}$:



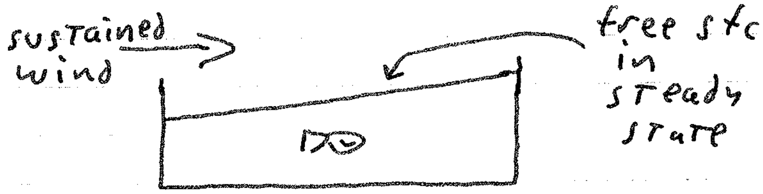
at $t = \frac{T}{2}$: $\eta = -2a\cos\frac{2\pi x}{2}$:



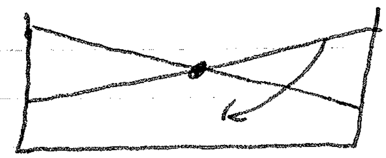
Standing surface gravity waves frequently occur in gulfs, channels, lakes, harbors and other laterally bounded fluid domains. They are also called <u>seiches</u>.

→ 1 handout: Lighthill's Fig. 60

Seiches can be generated by changes in wind stress at the free sfc:



Then wind dies. Now free sfc rocks back and forth



or generated by sudden changes in atmospheric pressure
" " " earthquakes
" " " tsunami (entering harbors)

Up to 17 nodes have been observed in seiches on large lakes (though 1 or 2 is most common).

Rare seiches can occasionally reach as high as 5–10 feet in amplitude and can damage ships, piers and wharves and erode beaches.

Seiches are very common on the Great Lakes. Usually small amplitude, but occasionally get extremely large amplitude + deadly seiches:

- Buffalo, NY (Lake Erie): 78 drowned on 18 Oct. 1844 by a 20–22 ft seiche.
- Grand Haven State Park (Lake Michigan): 10 drowned on 4 July 1929
- 100 mile stretch of Lake Erie: 7 drowned May 31, 1942
- Chicago, IL (Lake Michigan): 8 drowned by a 10 ft seiche on 26 June 1954

Resonance can occur ~~at~~ in case of forced seiches if frequency of forcing is close to one of the natural frequencies of the ~~harbor~~ basin.

e.g. Alaska earthquake of 1964 upset swimming pools as far away as Puerto Rico.

e.g. frequency of tsunami in Hilo, 1946, matched a natural frequency of the harbor. Got a devastating tsunami/seiche

Can also have internal standing waves (internal seiches):

e.g. seiching thermocline of lakes

e.g. atmospheric seiches along inversions in basins and valleys. A recent field experiment in Arizona's Meteor Crater was designed to study nocturnal boundary layer structure + dynamics in an environment relatively free of synoptic-scale impacts. Documented pronounced seiching driven by differential heating/cooling of the surface. [METCRAX field experiment, B.A.M.S. Nov 2008]