Some non-linear considerations

The simplest nonlinear p.d.e. in dynamics is the one-dimensional advection eq^n:

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

for $u = u(x,t)$.

The total derivative of u, $\frac{Du}{Dt}$ is the x-comp of the acceleration of an air parcel. Recall:

$$\frac{D()}{Dt} = \frac{\partial ()}{\partial t} + \vec{u}\cdot\nabla() \quad \text{[an identity]}$$

$$= \frac{\partial ()}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$\therefore$, since u is indep of y and z (or if we consider $v=0$, $w=0$):

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

$$\therefore \quad \underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}_{\substack{\text{Eulerian eq}^n \\ \text{of motion for} \\ \text{non-accelerating flow.}}} \quad \text{implies that} \quad \underbrace{\frac{Du}{Dt} = 0}_{\substack{\text{Lagrangian eq}^n \\ \text{of motion for} \\ \text{non-accelerating flow}}}$$

These eq^{ns} are <u>equivalent</u>.

Note that the Lagrangian eq^n $\frac{Du}{Dt} = 0$ is an <u>ordinary differential eq^n</u>. And it's <u>linear</u>! And we can integrate it right away, to get

<u>$u(t) = \text{const}$</u> for an <u>air parcel</u>.

But $u(t)$ is the x-comp velocity for an air parcel.

$$\therefore \quad \frac{dx}{dt} = u = \text{const} \quad \text{for an } \underline{\text{air parcel}}$$

$$\therefore \quad x = \text{const} \times t + \overset{\text{another}}{\text{const}}$$

This is the trajectory of the parcel. Applying the initial cond^{ns} $x(0) = x_0$, $u(0) = u_0$, we find that:

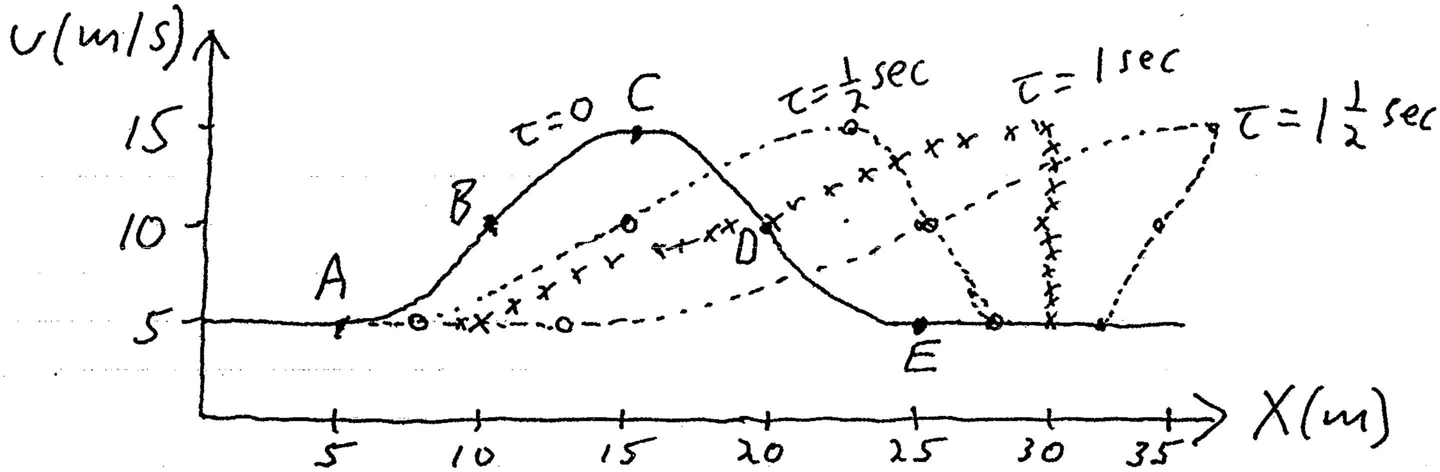
$$\left.\begin{aligned} u(t) &= u_0 \\ x(t) &= u_0 t + x_0 \end{aligned}\right\} \quad \begin{array}{l}\text{Solution} \\ \text{(in Lagrangian viewpoint)}\end{array}$$

In contrast, the <u>Eulerian description</u> of the exact same physical problem is a <u>partial differential eq^n</u> that's <u>non-linear</u>! In practice, certain types of ~~pd~~ nonlinear p.d.e.s (like this one) are solved by adopting a <u>Lagrangian</u> viewpoint and integrating along trajectories $\longrightarrow$ <u>Method of characteristics.</u>

$\frac{Du}{Dt} = 0 \quad \therefore u = \text{const}$ (for a parcel)

Associated with this simple sol^n is an interesting property: <u>nonlinear steepening</u> and <u>wave-breaking</u>.

Consider evolution of an initially symmetric hump-like disturbance:



A, B, C, D, E are air parcels. They have u-velocity components evident from the <u>initial profile</u> as:

$u_A = 5\,\text{m/s} \quad \therefore \quad X_A = X_A(0) + 5\tau = 5 + 5\tau$

$u_B = 10\,\text{m/s} \quad \therefore \quad X_B = X_B(0) + 10\tau = 10 + 10\tau$

$u_C = 15\,\text{m/s} \quad \therefore \quad X_C = X_C(0) + 15\tau = 15 + 15\tau$

$u_D = 10\,\text{m/s} \quad \therefore \quad X_D = X_D(0) + 10\tau = 20 + 10\tau$

$u_E = 5\,\text{m/s} \quad \therefore \quad X_E = X_E(0) + 5\tau = 25 + 5\tau$

	at $\tau = 1/2$ sec	$\tau = 1$ sec	$\tau = 1\,1/2$ sec
$X_A =$	7.5 m	10 m	12.5 m
$X_B =$	15 m	20 m	25 m
$X_C =$	22.5 m	30 m	37.5 m
$X_D =$	25 m	30 m	35 m
$X_E =$	27.5 m	30 m	32.5 m

Note how profile becomes asymmetric, Region where $\partial u/\partial x < 0$ becomes steeper. Eventually the solution becomes multivalued. This is called solution breakdown or wave breaking.

Analogous phenomenon: consider cars on a straight highway. Each car is moving at a const speed ($Du/Dt = 0$). Suppose previous diagram pertains to speed of 5 cars A, B, C, D, E. Get solution breakdown (car crash) at $t = 1$ sec at $x = 30$ m

Note that the solution is not physically reasonable at (or after) time of breakdown. Governing eqn is too simple. Some other physical process must be included (e.g. diffusion).

For the simple ~~system~~ eqn $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ $\left[\frac{Du}{Dt} = 0\right]$, the steepening effect is "inexorable". It always happens in regions where $\partial u/\partial x < 0$ (i.e. compressive region) and always leads to breaking.

Steepening can be countered (breaking prevented) if we include friction:

$$\frac{\partial u}{\partial t} + \frac{u\,\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad \text{(Burger's eqn)}$$

Or consider scenario where we neglect friction but include pressure gradient force and gravity and w (i.e. work with $\boxed{\text{Euler eqns}}$). We sort of did this in our study of gravity waves (in Adv. Dyn. I) but we linearized the top/interface kinematic + dynamical b.c.. If we had worked w/ full non-linear problem we would have found that: nonlinearity leads to breaking in shallow-water limit (which is non-dispersive) but not in deep-water limit.

Deep-water case is dispersive. Steepening by nonlinearity is balanced by spreading effect of dispersion. Can get periodic waves of unchanging form:

← periodic but not sinusoidal. Broad troughs, "peaky" crests.

Can also get isolated hump-like disturbances (solitons).

More nonlinear considerations: Solving problems with quadratic nonlinearities.

e.g. fall speed of a raindrop or hailstone in an atmosphere at rest.

$F = ma$ where F is sum of the forces on the drop (gravity and drag force), $a = dw/dt$ is acceleration, and m is mass of the drop, $m = \rho_d V_d$

$$\therefore \quad -mg + F_{drag} = m \frac{dw}{dt}$$

density of air

empirical drag force law: $F_{drag} = \rho_a \frac{\pi}{2} C_D R^2 w^2$

drag coefficient, radius

$\div$ by m

$$\frac{dw}{dt} = \frac{\rho_a \frac{\pi}{2} C_D R^2 w^2}{\rho_d V_d} - g \qquad V_d \to \frac{4}{3}\pi R^3$$

$$(*) \quad \therefore \frac{dw}{dt} = K w^2 - g \qquad \text{where } K \equiv \frac{3}{8} \frac{\rho_a}{\rho_d} \frac{C_D}{R}$$

This could be solved by separation of variables, but consider an alternative procedure (which would work even if K is time-dependent).

Eq^n (☆) is a special case of a <u>Riccati eq^n</u>:

$$\frac{df}{dx} + a(x) f^2 + b(x) = 0$$

[correspondence to our problem: $f \leftrightarrow W$, $x \leftrightarrow \tau$, $a \leftrightarrow -K$, $b \leftrightarrow g$]

A Riccati eq^n can be rewritten as a 2nd order <u>linear</u> eq^n by using the change of variable:

$$f = \frac{\alpha(x)}{\lambda} \frac{d\lambda}{dx} \qquad [\alpha = ?, \ \lambda = ?]$$

$$\therefore \frac{df}{dx} = \frac{d\alpha}{dx} \frac{1}{\lambda} \frac{d\lambda}{dx} - \frac{\alpha}{\lambda^2}\left(\frac{d\lambda}{dx}\right)^2 + \frac{\alpha}{\lambda} \frac{d^2\lambda}{dx^2}$$

Plug into the Riccati eq^n:

$$\frac{d\alpha}{dx} \frac{1}{\lambda} \frac{d\lambda}{dx} \boxed{- \frac{\alpha}{\lambda^2}\left(\frac{d\lambda}{dx}\right)^2} + \frac{\alpha}{\lambda} \frac{d^2\lambda}{dx^2} + \boxed{a \frac{\alpha^2}{\lambda^2}\left(\frac{d\lambda}{dx}\right)^2} + b = 0$$

in order for these 2 terms to cancel, must have

$$a\alpha^2 - \alpha = 0$$

$$\therefore \boxed{\alpha = \frac{1}{a}}$$ this defines α

$$\therefore -\frac{1}{a^2} \frac{da}{dx} \frac{1}{\lambda} \frac{d\lambda}{dx} + \frac{1}{a\lambda} \frac{d^2\lambda}{dx^2} + b = 0$$

mult by $a\lambda$

$$\frac{d^2\lambda}{dx^2} - \frac{1}{a}\frac{da}{dx}\frac{d\lambda}{dx} + ab\lambda = 0$$

This is a 2^{nd} order linear o.d.e!

Applying this procedure to our fall speed problem, we get:

$$w = -\frac{1}{k}\frac{1}{\lambda}\frac{d\lambda}{d\tau} \quad \text{where} \quad \frac{d^2\lambda}{d\tau^2} - kg\lambda = 0$$

sol^n: try $\lambda = e^{m\tau}$

$$\therefore m^2 - kg = 0$$
$$m^2 = kg$$
$$m = \pm\sqrt{kg}$$

$$\therefore \lambda = c_1 e^{\sqrt{kg}\,\tau} + c_2 e^{-\sqrt{kg}\,\tau}$$

$$\therefore w = -\frac{1}{k}\frac{\left[c_1\sqrt{kg}\,e^{\sqrt{kg}\,\tau} - c_2\sqrt{kg}\,e^{-\sqrt{kg}\,\tau}\right]}{c_1 e^{\sqrt{kg}\,\tau} + c_2 e^{\sqrt{kg}\,\tau}}$$

factor out c_1 from top and bottom

$$= -\sqrt{\frac{g}{k}}\;\frac{e^{\sqrt{kg}\,\tau} - \beta e^{-\sqrt{kg}\,\tau}}{e^{\sqrt{kg}\,\tau} + \beta e^{-\sqrt{kg}\,\tau}}$$

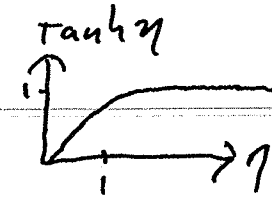
where $\beta \equiv \frac{c_2}{c_1}$ is some const.

Initial cond^n: $w(0) = 0$

$$\therefore 0 = -\sqrt{\frac{g}{k}}\;\frac{1-\beta}{1+\beta} \qquad \therefore \beta = 1$$

$$\therefore w = -\sqrt{\frac{g}{k}}\,\tanh\left(\sqrt{kg}\,\tau\right)$$

recall:



So terminal velocity is: $w_T = -\sqrt{\frac{g}{k}} = -\sqrt{\frac{8}{3}\frac{\rho_d}{\rho_a}\frac{R}{C_D} g}$

and it's reached (roughly) at times when $\sqrt{kg}\,\tau > 1$

i.e. for $\tau > \frac{1}{\sqrt{kg}} = \sqrt{\frac{8}{3}\frac{\rho_d}{\rho_a}\frac{R}{C_D g}}$.

So, larger terminal velocities (and larger times to get terminal velocity) for larger drops and higher altitudes (smaller ρ_a).

Now let's look at Burger's equation:

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

$$u\frac{\partial u}{\partial x} \to \frac{\partial}{\partial x}\frac{u^2}{2}$$

It looks like we can almost integrate it w.r.t. x, but trouble with the first term. So introduce a change of variable:

$$(1) \quad u = \frac{\partial \phi}{\partial x} \quad \text{defines } \phi \quad \left[\phi \equiv \int_0^x u\,dx'\right]$$

but for now just apply it in first term:

$$\frac{\partial}{\partial \tau}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\frac{u^2}{2} = \nu\frac{\partial^2 u}{\partial x^2}$$

$$\therefore \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial \tau} + \frac{u^2}{2} - \nu\frac{\partial u}{\partial x}\right] = 0 \quad \text{int w.r.t. } x$$

$$(2) \quad \frac{\partial \phi}{\partial \tau} + \frac{u^2}{2} - \nu\frac{\partial u}{\partial x} = F(t)$$

← an irrelevant function of integration. Set it to 0 [could always redefine ϕ: $\phi_{new} = \phi_{old} - \int^{\tau} F(\tau')\,d\tau'$ which removes $F(t)$ from (2) but leaves x-deriv of ϕ unchanged so u is unchanged]

If we forget about 1st term, (2) "looks like" a Ricatti eq^n. Will the same "Riccati trick" (linearizing transformation) work in this case?

Try it and see! $u = \frac{\alpha}{\lambda}\frac{\partial \lambda}{\partial x}$ [$\alpha = ?$, $\lambda = ?$]

$$\therefore \quad \boxed{u = \alpha \frac{\partial}{\partial x} \ln \lambda}$$

However we've already introduced $u = \frac{\partial \phi}{\partial x}$

$\therefore$ as long as we take α to be a constant, we see that

$$\boxed{\phi = \alpha \ln \lambda}$$

and so (2) now becomes

$$\frac{\alpha}{\lambda}\frac{\partial \lambda}{\partial \tau} + \frac{\alpha^2}{2}\frac{1}{\lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2 - \nu \frac{\alpha}{\lambda}\frac{\partial^2 \lambda}{\partial x^2} + \frac{\nu \alpha}{\lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2 = 0$$

choose $\frac{\alpha^2}{2} + \nu\alpha = 0 \rightarrow \boxed{\alpha = -2\nu}$

$$\therefore \quad -\frac{2\nu}{\lambda}\frac{\partial \lambda}{\partial \tau} + \frac{2\nu^2}{\lambda}\frac{\partial^2 \lambda}{\partial x^2} = 0 \qquad \text{mult by } \frac{\lambda}{2\nu}$$

$$\boxed{\frac{\partial \lambda}{\partial \tau} = \nu \frac{\partial^2 \lambda}{\partial x^2}}$$

Linear 2nd order p.d.e.
It's the heat eqn
(a.k.a. diffusion eqn).

where $\boxed{u = -\frac{2\nu}{\lambda}\frac{\partial \lambda}{\partial x}}$

Many analytical solns of the heat eqn have been obtained. They can be used to generate analytical solns of Burger's eqn. These solns are used as benchmark (validation) solutions in Computational Fluid Dynamics. [The transformation $u = -\frac{2\nu}{\lambda}\frac{\partial \lambda}{\partial x}$ is the Cole-Hopf transformation. Discovered independently by Cole and Hopf].

→ Look at Benton + Platzman paper (handout)