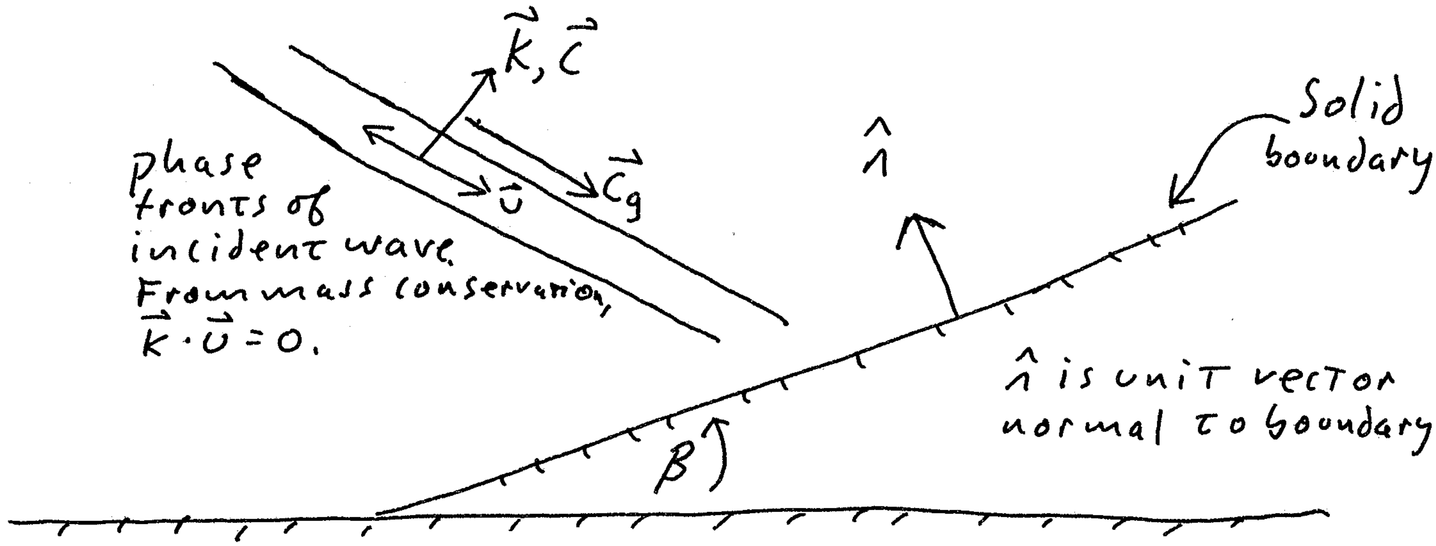
Reflection of small-scale internal gravity waves from a sloping solid boundary (e.g. mountain range).



Put origin of coord system on the bdry. Then for any location $\vec{x}$ on the boundary:

$$\boxed{\vec{x}\cdot\vec{n}=0} \text{ for } \vec{x} \text{ on the boundary.}$$

What is the nature of the wave reflection?

The whole disturbance is a linear superposition (sum) of the incident wave and the reflected wave (assuming the waves behave linearly → "small amplitude").

∴ The impermeability cond^n becomes:

sum of incident and reflected waves must have 0 normal velocity at all points on the boundary at all times.

Velocity associated w/ incident wave:

$$\vec{u}_I = \vec{\alpha}_I \cos(\vec{k}_I\cdot\vec{x} - \omega_I t)$$

where $\vec{\alpha}_I$ is a vector velocity amplitude.

Velocity associated with <u>refected wave</u>:

$$\vec{U}_R = \vec{\alpha}_R \cos(\vec{k}_R \cdot \vec{x} - \omega_R t + \delta)$$

where δ is any possible phase shift of reflected wave w.r.t. incident wave.

So impermeability cond^n becomes:

$$\hat{n} \cdot (\vec{U}_I + \vec{U}_R) = 0 \quad \underline{\text{on boundary}}.$$

$$\therefore \quad \hat{n} \cdot \vec{\alpha}_I \cos(\vec{K}_I \cdot \vec{x} - \omega_I t)$$
$$+ \hat{n} \cdot \vec{\alpha}_R \cos(\vec{k}_R \cdot \vec{x} - \omega_R t + \delta) = 0 \quad \text{on boundary}$$

now look at this term

We can always decompose a wavenumber into normal and tangential comp^s; so:

$$\vec{K}_I = \vec{K}_{In} + \vec{K}_{Is}$$

$$\vec{k}_R = \vec{k}_{Rn} + \vec{k}_{Rs}$$

"n" = normal
"s" = tangential

So, on the boundary:

$$\vec{K}_I \cdot \vec{x} = \underbrace{\vec{K}_{In} \cdot \vec{x}}_{} + \vec{K}_{Is} \cdot \vec{x} = \vec{K}_{Is} \cdot \vec{x}$$

This is 0 on bdry since $\vec{K}_{In}$ is in $\hat{n}$ dir^n, and $\hat{n} \cdot \vec{x} = 0$

similarly, $\vec{K}_R \cdot \vec{x} = \vec{k}_{Rs} \cdot \vec{x}$. Plug in to impermeability:

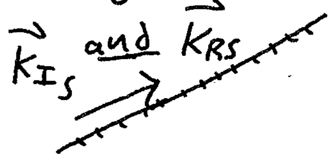
$$(\ast) \quad \begin{aligned} &\hat{n} \cdot \vec{\alpha}_I \cos(\vec{k}_{Is} \cdot \vec{x} - \omega_I t) \\ &+ \hat{n} \cdot \vec{\alpha}_R \cos(\vec{k}_{Rs} \cdot \vec{x} - \omega_R t + \delta) = 0 \quad \underline{\text{on bdry}}. \end{aligned}$$

In order for (✪) to be satisfied at all points $\vec{x}$ on the boundary and at all times, must have:

(1) $\delta = 0$

(2) $\omega_I = \omega_R$ [freq of incident/reflected waves same]

(3) $\vec{K}_{I_S} = \vec{K}_{RS}$ [amount of wavenumber vector directed along boundary is the same]

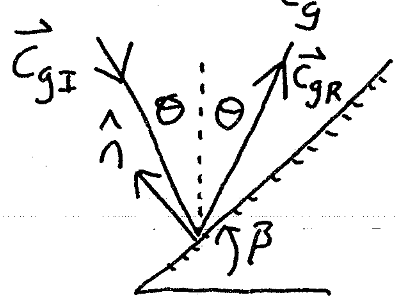


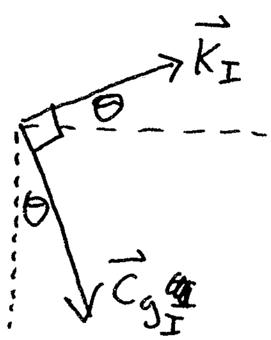
(4) $\vec{\alpha}_I \cdot \hat{n} = -\vec{\alpha}_R \cdot \hat{n}$ [comp of velocity amplitude normal to bdry is equal and opposite].

Statement (2) $\omega_I = \omega_R$ means $N\cos\theta_I = N\cos\theta_R$

$\therefore \cos\theta_I = \cos\theta_R$

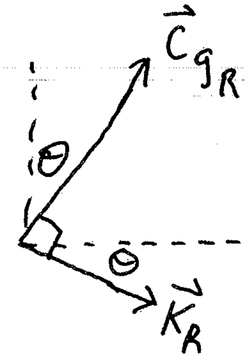
where θ is angle that $\underline{\vec{k} \text{ (and } \vec{c}\text{)}}$ make w.r.t. $\underline{\text{horiz}}$.
equivalently, θ " " $\vec{c}_g$ makes w.r.t. $\underline{\text{vertical}}$.





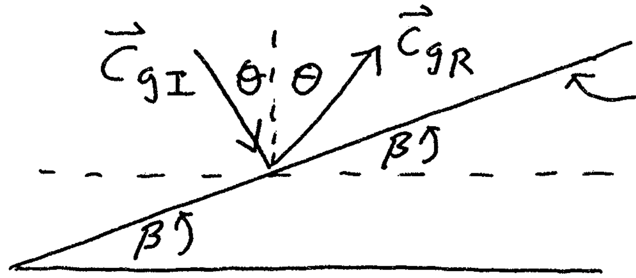
incident wave

reflected wave



$\underline{\text{Not}}$ angle of incidence = angle of reflection.

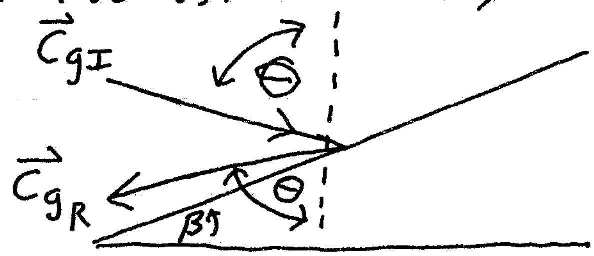
Now have: angle of $\vec{k}, \vec{c}$ w.r.t. horiz } same before and after reflection
angle of $\vec{c}_g$ w.r.t. vert }



boundary makes an angle β w.r.t. horiz, $90° - \beta$ w.r.t. vertical.

Cond^n for reflected wave to be parallel to boundary [$\vec{C}_{gR}$ tangent to bdry] is that $\theta = 90° - \beta$

If $\underline{\theta > 90° - \beta}$ then waves can't reflect forward (as drawn above). Instead, they reflect <u>backward</u>:



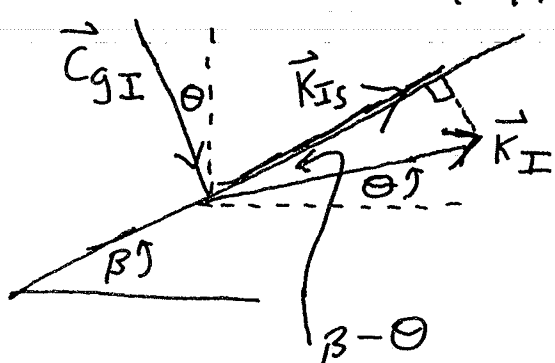
θ "big" in sense that $\theta > 90° - \beta$ is associated w/ backward reflections.

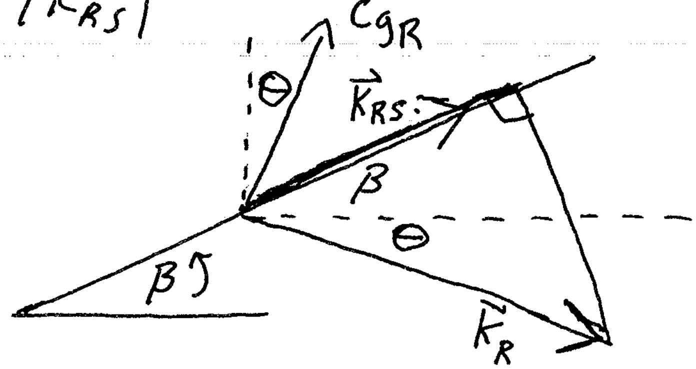
Get backward reflections for steep slopes (β large) or low frequency waves [since $\omega = N \cos\theta$ then, for fixed N, if ω is small then $\cos\theta$ is small $\therefore \theta$ is big]

Now look at implications of statement (3) [we'll examine it for forward reflections, but get same result for back reflec.]

$$\vec{K}_{Is} = \vec{K}_{Rs}$$

$$\therefore |\vec{K}_{Is}| = |\vec{K}_{Rs}|$$





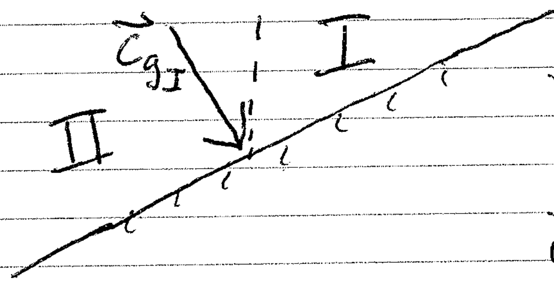
$\therefore |\vec{K}_{Is}| = K_I \cos(\beta - \theta)$ where $K_I = |\vec{K}_I|$

$|\vec{K}_{Rs}| = K_R \cos(\theta + \beta)$ where $K_R = |\vec{K}_R|$

Previous formulas $|\vec{K}_{I_S}| = K_I \cos(\beta - \theta)$

$$|\vec{K}_{R_S}| = K_R \cos(\theta + \beta)$$

were obtained for the scenario where:



incoming wave is in region II. If incoming wave was in region I the formulas would be somewhat different [and result would be different]

So, since $|\vec{K}_{I_S}| = |\vec{K}_{R_S}|$

$$\therefore \quad K_I \cos(\beta - \theta) = K_R \cos(\theta + \beta)$$

$$\therefore \quad K_I (\cos\beta \cos\theta + \sin\beta \sin\theta) = K_R (\cos\theta \cos\beta - \sin\theta \sin\beta)$$

$$\therefore \quad K_R = K_I \underbrace{\left(\frac{\cos\beta \cos\theta + \sin\beta \sin\theta}{\cos\beta \cos\theta - \sin\beta \sin\theta} \right)}_{\text{behaves as } \frac{a+b}{a-b} \text{ where } a>0,\ b>0}$$

$\therefore K_R > K_I$

$\therefore$ Wavenumber magnitude <u>increases</u> upon reflection (for incoming waves in region II)

$\therefore \lambda$ decreases $\therefore$ <u>shear</u> increases.

Biggest possible value of K_R occurs when $\theta + \beta = 90^\circ$ [$\vec{C}_{g_R}$ tangent to bdry]

Then: $K_I \underbrace{\cos(\beta-\theta)}_{\text{not } 0} = K_R \underbrace{\cos(\theta+\beta)}_{\to 0}$

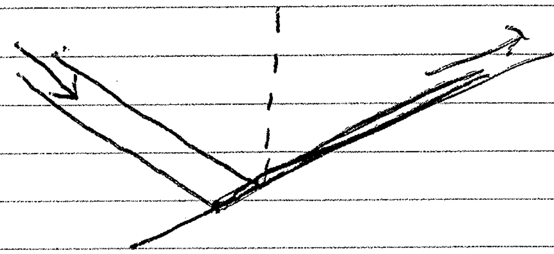
So, since l.h.s. is finite and nonzero must have $K_R \to \infty$

∴ $\lambda \to 0$

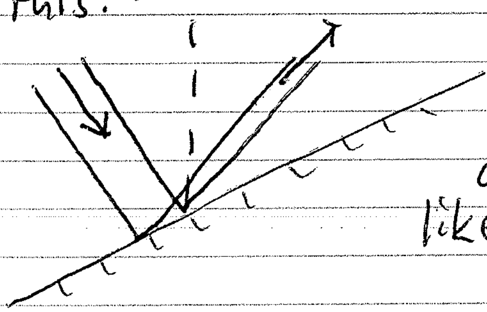
∴ very high shear

∴ get turbulence then frictional dissipation

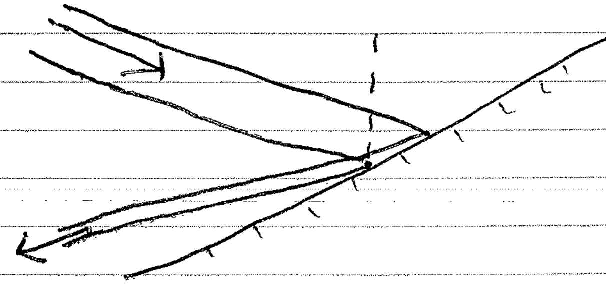
∴ wave energy gets absorbed.



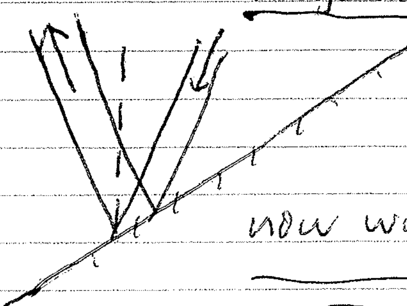
In general, for incoming waves in region II, beam reflects like this:



or like this:



while incoming waves in region I reflect like this:



now wavelength ↑ upon reflection!

lastly, mentioned that in oceans, near coasts, can get backward reflections + forward reflection + wave energy absorption + high shear + turbulence, etc: