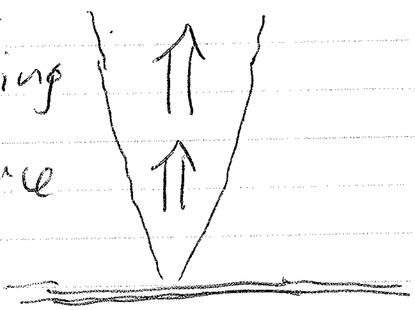
# Prototypical natural convection flows: Plumes and Thermals

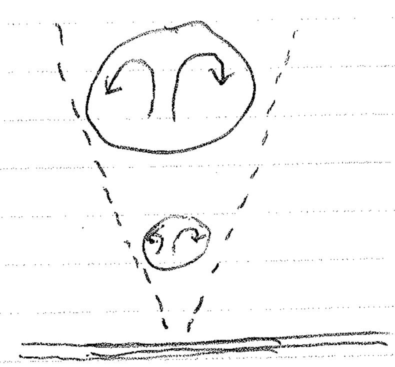
Some good references:

Batchelor 1954 QJRMS
Morton, Taylor, and Turner 1956 Proc. Roy. Soc. London, A
Emanuel's textbook on convection
Turner's book on Buoyancy Effects in Fluids
Fannelöp's book: Fluid Mechanics for Industrial Safety and Environmental Protection, 2012

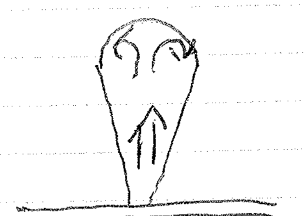
Plumes: Spatially continuous buoyant regions arising from a maintained point, line, or areal source of buoyancy.



Thermals: Discrete buoyant elements that arise from an impulsive or otherwise unsteady source of buoyancy.



Starting Plume: A plume in its development stage.

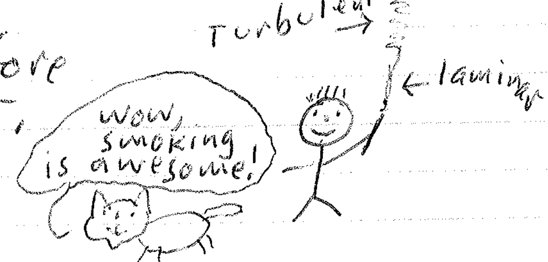


The simplest types of these flows arise from point or line singularities. Can be studied in the lab, in numerical experiments and in theoretical analyses (similarity models combined with the Pi Theorem).

The main interest in these idealized flows is as "building blocks" in cumulus entrainment and as prototypical flow structures in the dry convective atmospheric boundary layer.

$$\boxed{\text{Batchelor's theory for a laminar plume over a maintained point source of heat in an unstratified environment.}}$$

e.g. cigarette plume before it becomes turbulent.




- Consider statistical steady state
- Consider variations in temp T to be <u>much</u> smaller than T itself (so we'll invoke the Boussinesq approx).
- Consider atmosphere to be <u>unstratified</u>

Governing eq^ns:

<u>Mass conservation</u> (well, incompressibility cond^n) : $\nabla \cdot \vec{u} = 0$

<u>Eq^n of motion</u>

perturbation pressure

$$(\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho_0}\nabla p + \left(\frac{T - T_0}{T_0}\right) g\hat{k} + \nu \nabla^2 \vec{u}$$

const density      const temp

<u>Heat eq^n</u> (Thermal energy eq^n) : $\vec{u} \cdot \nabla T = \kappa \nabla^2 T$

Can rewrite heat eq$^n$ in terms of the same buoyancy-like variable that appears in eq$^{n}$ of motion:

$$\vec{v} \cdot \nabla\left[\left(\frac{T - T_0}{T_0}\right) g\right] = \kappa \nabla^2\left[\left(\frac{T - T_0}{T_0}\right) g\right]$$

Look for <u>approximate solutions</u> in the form of <u>similarity solutions</u> (self-similarity solutions), i.e. solutions that look the same on every horizontal cross-section:

other parameters of the problem (e.g. $\nu$ and $\kappa$) are also in here →

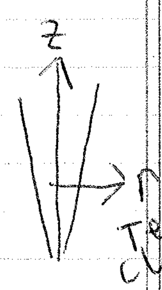
$$\left\{\begin{aligned} w &= z^m \times \text{function}\,(r/R) \\ g\left(\frac{T - T_0}{T_0}\right) &= z^n \times \text{another function}\,(r/R) \\ R &= C z^l \quad \text{where } C \text{ is some constant} \end{aligned}\right.$$

Note: Proposing power laws in $z$ goes beyond the similarity assumption, i.e. can look for height dependencies that are more general than power laws.

Note: We can interpret $R$ as a measure of plume radius.

Consider <u>axisymmetric flow</u> due to a point source and assume that the diffusion term is dominated by changes in the <u>radial direction</u>, not the vertical dir$^n$, so:

$$\nabla^2(\,) = \underbrace{\frac{\partial^2}{\partial z^2}(\,)}_{\text{neglect}} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}(\,) + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}(\,)}_{0 \text{ since flow is axisymmetric}}$$




Temp and $\vec{v}$ change more rapidly in $r$ direction than $z$ direction

neglect
It's an <u>internal</u> <u>boundary layer approximation.</u>

0 since flow is axisymmetric

So $\nabla^2() \approx \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}()$

So now the steady state heat eq$^n$ becomes:

$$\vec{v}\cdot\nabla\left[\left(\frac{T-T_0}{T_0}\right)g\right] = \frac{\kappa}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\left[\left(\frac{T-T_0}{T_0}\right)g\right]$$

rewrite l.h.s. as: $\nabla\cdot\left[\vec{v}\left(\frac{T-T_0}{T_0}\right)g\right]$

scratch paper

for any scalar $\phi$: $\nabla\cdot(\vec{v}\phi) = \phi\nabla\cdot\vec{v} + \vec{v}\cdot\nabla\phi$

and since our flow is <u>incompressible</u> ($\nabla\cdot\vec{v}=0$):

$\nabla\cdot(\vec{v}\phi) = \vec{v}\cdot\nabla\phi$

Integrate this heat eq$^n$ over a slab-like volume btw 2 z levels extending from $r=0$ to $r=\infty$:

$$\int_V \nabla\cdot\left[\vec{v}\left(\frac{T-T_0}{T_0}\right)g\right]dV = 2\pi\kappa\iint\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\left[\left(\frac{T-T_0}{T_0}\right)g\right] r\,dr\,dz$$

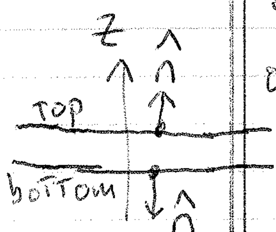
These cancel!

where $dV = 2\pi r\,dr\,dz$

use this explicitly in the rhs integral

Use divergence th$^m$ on l.h.s., get:

$$\underbrace{\int_A \vec{v}\left(\frac{T-T_0}{T_0}\right)g\cdot\hat{n}\,dA}_{} = 2\pi\kappa\underbrace{\iint\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\left[\left(\frac{T-T_0}{T_0}\right)g\right]dr\,dz}_{\text{can integrate w.r.t. } r}$$




on lateral sides (at $r\to\infty$): $T\to T_0$ and $\vec{v}\to 0$

so lateral integral drops out.

Only top/bottom area integrals survive.

$$\therefore \int_{\substack{\text{Top}\\ \text{area}}} w\left(\frac{T-T_0}{T_0}\right) g\, dA - \int_{\substack{\text{bottom}\\ \text{area}}} w\left(\frac{T-T_0}{T_0}\right) g\, dA$$

$$= 2\pi k \int \left[ r\frac{\partial}{\partial r}\left(\frac{T-T_0}{T_0}\right) g \right]\Bigg|_{r=0}^{r=\infty} dz$$

The quantity [...] on the r.h.s. vanishes at $r=0$ (because of the $r$ in it). If we assume that $T \to T_0$ sufficiently rapidly as $r \to \infty$ (e.g. if $T - T_0$ decays exponentially with $r$) then the r.h.s. $\to 0$.

$$\therefore \int_{\substack{\text{Top}\\ \text{area}}} w\left(\frac{T-T_0}{T_0}\right) g\, dA = \int_{\substack{\text{bottom}\\ \text{area}}} w\left(\frac{T-T_0}{T_0}\right) g\, dA$$

but the top and bottom are at arbitrary heights so really we have:

$$\int_{\substack{\text{any}\\ \text{infinite}\\ \text{horiz area}}} w\left(\frac{T-T_0}{T_0}\right) g\, dA = \text{const, indep of } z$$

or, since $dA = 2\pi r\, dr$:

$$2\pi \int_0^\infty w g \left(\frac{T-T_0}{T_0}\right) r\, dr = \underset{\substack{\uparrow\\ \text{call it } F}}{\text{const, indep of } z}$$

Can interpret $F$ as the heat flux, the rate at which heat is released by the source.

Rewrite this slightly so we can plug in our similarity expressions:

$$2\pi \int_0^\infty w g \left(\frac{T-T_0}{T_0}\right) R^2 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = F$$

Now plug in the expressions for $w$, $g\left(\frac{T-T_0}{T_0}\right)$, $R$:

$$2\pi C^2 z^m z^{2l} z^n \int_0^\infty \text{function}\left(\frac{r}{R}\right) \times \begin{array}{c}\text{another}\\ \text{function}\end{array}\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = F$$

This definite integral is just some number. No $z$-dependence (or even $r$ dependence) in it.

And no $z$-dependence on r.h.s. So no $z$-dependence on l.h.s. so

$$\therefore \boxed{m + 2l + n = 0}$$

This is a constraint on the power laws.

Get more constraints on the power laws by working with the vertical comp of the eq$^n$ of motion

$$\underset{\text{accel}^n}{w \frac{\partial w}{\partial z}} \sim \underset{\text{buoyancy}}{g \frac{(T-T_0)}{T_0}} \sim \underset{\text{friction}}{\frac{\nu}{r} \frac{\partial}{\partial r} r \frac{\partial w}{\partial r}} \qquad r = R\left(\frac{r}{R}\right)$$

$$\frac{1}{R^2} \frac{\partial}{\partial(r/R)} \frac{\partial}{\partial(r/R)} \left(\frac{r}{R}\right) \frac{\partial w}{\partial(r/R)}$$

just focus on the $z$-dependence, get

$$z^{2m-1} \sim z^n \sim z^{m-2l}$$

$$\therefore 2m - 1 = n = m - 2l$$

Together with other constraint for $m, n, l$, get 3 eq$^{ns}$ for 3 unknowns.

$n = m - 2l$ plug into $m + 2l + n = 0$

$\therefore m + 2l + m - 2l = 0$

$\therefore 2m = 0$

$\therefore \boxed{m = 0}$

Then use $2m - 1 = n$

$\therefore n = -1$

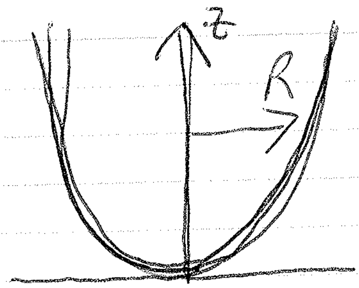
lastly $\underset{-1}{n} = \underset{0}{m} - 2l \quad \therefore l = 1/2$

So: $\boxed{w = \text{function}\left(\frac{r}{R}\right)}$ (involving other governing parameters but NOT $z$!)

$\boxed{g\left(\frac{T - T_0}{T_0}\right) = \frac{1}{z}\,\text{function}\left(\frac{r}{R}\right)}$ " "

$\boxed{R = c\sqrt{z}}$ " "

So plume looks like a parabola $(z \sim R^2)$




Now, to see how the other parameters enter the solution, use the Pi Theorem.