Laminar plume over a point source of heat in an unstratified environment (continued)

To see how the other parameters enter the solution, use the Pi Theorem.

First look at the expression for R: $R = c\sqrt{z}$

R and z are coupled in the combination $\frac{R}{\sqrt{z}}$. Keep them together like this.

$\frac{R}{\sqrt{z}} = c$ ← const that can vary with other const parameters: F, ν, κ

Look at dimensions:

$$\left[\frac{R}{\sqrt{z}}\right] = L^{1/2}$$

$$[\nu] = \frac{L^2}{T}$$

$$[\kappa] = \frac{L^2}{T}$$

Temp, not Time

$$[F] = \left[2\pi \int_0^\infty w g \frac{(T - T_0)}{T_0} r\,dr\right] = \frac{L}{T}\frac{L}{T^2}L^2 = \frac{L^4}{T^3}$$

We have 4 parameters $\frac{R}{\sqrt{z}}, \nu, \kappa, F$ $(m = 4)$

2 dimensions L, T. Can show that k is also 2.

e.g. F and κ cannot be combined to get a non-dim quantity. But every set of 3 can be combined to get a non-dim combo.

$n = m - k$

So the pi number is: $m - k = 4 - 2 = 2$.

So 2 Pi groups.

One obvious pi group is $\Pi_1 \equiv \frac{\nu}{\kappa}$ [Prandtl number $\sigma = \frac{\nu}{\kappa}$]

Just need to find one more Pi group. It should include the thing we want $R/\sqrt{z}$ and make sure to use what hasn't been used yet. Try:

$$\Pi_2 = \frac{R}{\sqrt{z}} \nu^a F^b$$

instead of ν could have used κ. Can also use both ν and κ but will get an underdetermined system from that (so no problem)

$$[\Pi_2] = \left[\frac{R}{\sqrt{z}} \nu^a F^b \right]$$

$$= L^{1/2} \left(\frac{L^2}{T}\right)^a \left(\frac{L^4}{T^3}\right)^b$$

$$= L^{\frac{1}{2}+2a+4b} \; T^{-a-3b}$$

dimensionless so these exponents must sum to 0

$\therefore \frac{1}{2} + 2a + 4b = 0$ and $-a - 3b = 0 \rightarrow a = -3b$

$\therefore \frac{1}{2} + 2(-3b) + 4b = 0$

$\therefore \frac{1}{2} - 2b = 0$

$\therefore \boxed{b = \frac{1}{4}}$ and $\boxed{a = -\frac{3}{4}}$

$$\therefore \Pi_2 = \frac{R}{\sqrt{z}} \nu^{-3/4} F^{1/4}$$

Pi Thm says: $\Pi_2 = f(\Pi_1)$

$$\therefore \frac{R}{\sqrt{z}} \nu^{-3/4} F^{1/4} = f(\sigma)$$

$$\therefore \quad \boxed{R = \sqrt{z}\ \frac{\nu^{3/4}}{F^{1/4}} \times \text{function}(\sigma)}$$

more heat input, the narrower the plume
The less viscous the fluid, the narrower the plume.

Now look at the buoyancy term, which we've already established varies as:

$$g\frac{(T-T_0)}{T_0} = \frac{1}{z}\,\text{function}\left(\frac{r}{R}\right)$$ and of the other parameters

$\therefore$ buoyancy and z appear in the combination $\left(z\ \frac{g(T-T_0)}{T_0}\right)$. It's a function of $\frac{r}{R}$ and ν, κ, F

5 parameters total ($m = 5$)

check dimensions:

$$\left[zg\left(\frac{T-T_0}{T_0}\right)\right] = L\frac{L}{T^2} = \frac{L^2}{T^2}$$

$$[\nu] = [\kappa] = L^2/T$$

$$[F] = L^4/T^3$$

$[r/R] = 1$ (well, it's non-dimensional)

2 dimensions (L, T) ... Can verify $k = 2$

$n = m - k = 5 - 2 = 3$ (pi number) So 3 pi groups.

As before, an obvious pi group is the Prandtl number, $\Pi_1 = \nu/\kappa$ (call it σ)

A second obvious pi group is $\Pi_2 \equiv \frac{r}{R}$ (it's already non-dim)

Can rewrite it as: $\Pi_2 = \frac{r}{\sqrt{z}} \frac{F^{1/4}}{\nu^{3/4}}$

The third (last) Pi group should include the thing we want (buoyancy) since it hasn't been included in the other Pi groups.

$$\Pi_3 \equiv z g \left(\frac{T-T_0}{T_0}\right) \nu^a F^b$$

$$[\Pi_3] = L \frac{L}{T^2} \left(\frac{L^2}{T}\right)^a \left(\frac{L^4}{T^3}\right)^b = L^{2+2a+4b} T^{-2-a-3b}$$

dimensionless

$\therefore \quad 2+2a+4b=0 \longleftarrow$ plug in

$-2-a-3b=0 \rightarrow a=-2-3b$

$\therefore \quad 2+2(-2-3b)+4b=0$

$\therefore \quad 2-4-6b+4b=0$

$\therefore \quad -2-2b=0 \quad \therefore \boxed{b=-1} \quad \therefore a=1$

So $\Pi_3 = z g \left(\frac{T-T_0}{T_0}\right) \frac{\nu}{F}$

Pi Th^m says $\Pi_3 = \text{function}(\Pi_1, \Pi_2)$

$$\therefore \quad z g \left(\frac{T-T_0}{T_0}\right) \frac{\nu}{F} = \text{function}\left(\sigma, \frac{r F^{1/4}}{\sqrt{z}\, \nu^{3/4}}\right)$$

or: $$\boxed{g \frac{(T-T_0)}{T_0} = \frac{F}{\nu z} \times \text{function}\left(\sigma, \frac{r F^{1/4}}{\sqrt{z}\, \nu^{3/4}}\right)}$$

Note that centerline ($r=0$) buoyancy decreases rapidly with distance above the source ($1/z$) and increases linearly with source strength F.

Lastly, examine w. We've already shown that:

$$w = z^{0} \times \text{function}(r/R) \quad 0 \leftarrow 0!$$

$\therefore$ list of param: $w, \frac{r}{R}, \nu, \kappa, F$ $(m=5)$ 2 dimensions (L, T)

$5-2=3$ pi number $=3$ (3 pi groups)

$$\Pi_1 \equiv \sigma = \nu/\kappa$$

$$\Pi_2 \equiv r/R$$

$$\Pi_3 = w\,\nu^a F^b$$

So $$[\Pi_3] = \frac{L}{T}\left(\frac{L^2}{T}\right)^a\left(\frac{L^4}{T^3}\right)^b = L^{1+2a+4b}\,T^{-1-a-3b}$$

$\therefore$ $1+2a+4b=0$, $-1-a-3b=0$ $\longrightarrow$ add 'em up, get $a+b=0 \rightarrow a=-b$

$\therefore -1-(-b)-3b=0$
$-1-2b=0$ $\therefore$ $\boxed{b=-\frac{1}{2}}$ $\boxed{a=\frac{1}{2}}$

$$\therefore \Pi_3 = w\frac{\sqrt{\nu}}{\sqrt{F}}$$

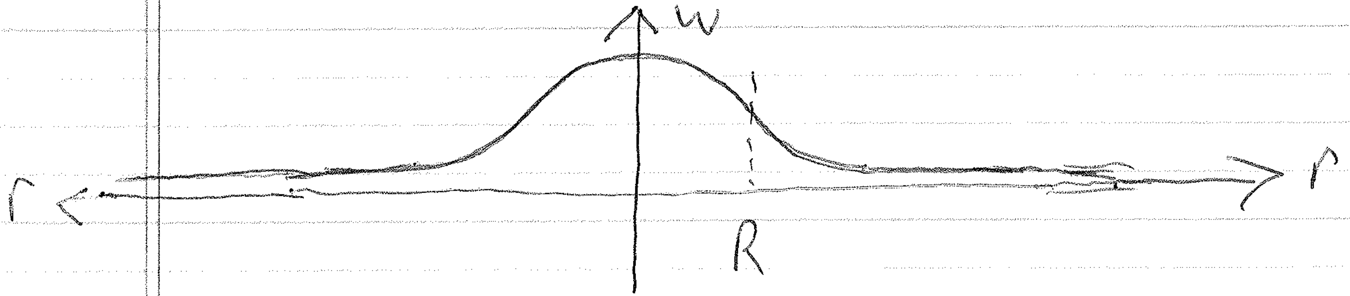
Pi Th^m says: $\Pi_3 = \text{function}(\Pi_1, \Pi_2)$

$$\therefore w\frac{\sqrt{\nu}}{\sqrt{F}} = \text{function}(\sigma, r/R)$$

$$\therefore \boxed{w = \frac{\sqrt{F}}{\sqrt{\nu}} \times \text{function}\left(\sigma, \frac{rF^{1/4}}{\sqrt{z}\,\nu^{3/4}}\right)}$$

$\therefore$ Along centerline $(r=0)$, w is independent of z, decreases with increasing viscosity, and increases with increasing source strength F!

Note: Yih (1951) and others have solved the differential eq^n for the functions of r/R. w looks like a Gaussian (on any horiz cross-section):



Note: Can define a Reynolds number for this flow,

$$Re \equiv \frac{w_{centerline} R}{\nu} = \frac{\sqrt{F}}{\sqrt{\nu}} \frac{\sqrt{z}\,\nu^{3/4}}{F^{1/4}\,\nu} \times f''(\sigma)$$

$$= \sqrt{z}\,\frac{F^{1/4}}{\nu^{3/4}} \times f''(\sigma)$$

which increases with z. In general, large Re $\rightarrow$ turbulence.

Laboratory and theoretical studies confirm that when Re exceeds a threshold value the laminar plume breaks down into a turbulent flow.

So, for z large enough, plume becomes turbulent.