Batchelor's theory for a turbulent plume over a maintained point source of heat in an unstratified environment.

- Again consider unstratified flow
  - flow is statistically steady
  - work with mean (ensemble) averages

For turbulent plumes ν and κ are irrelevant, (except right near the source). So neglect ν and κ.

Again consider similarity solutions but this time the problem (with no ν or κ) is so simple that dimensional analysis will give the z-depencies. No need to assume/hypothesize power laws in z.

Again, we'll work with the heat equation. But this time the diffusion (κ) term is neglected. Again, integrate it over a slab volume. Find that the heat flux is again independent of height (although this time it's composed of two parts, a mean flux and a turbulent covariance contribution).

ensemble ave of w

So, to get the mean $\overline{w}$ profile, consider:

"list of parameters": $\overline{w}$, r/R, F, z $\quad m=4$

dimensions: L/T, 1, L^4/T^3, L

2 dimensions $\quad (k=2)$

$n = m - k = 4 - 2 = 2$ pi numbers

Clearly one of the pi groups can be taken to be:

$$\Pi_1 \equiv r/R$$

Next, consider $\Pi_2 = \bar{w}\, z^a F^b$

$$[\Pi_2] = \frac{L}{T} L^a \left(\frac{L^4}{T^3}\right)^b = L^{1+a+4b}\, T^{-1-3b}$$

$\therefore$ $1 + a + 4b = 0$

and $-1 - 3b = 0 \rightarrow b = -1/3$

$\therefore$ $1 + a - \frac{4}{3} = 0 \rightarrow a = 1/3$

$\therefore$ $\Pi_2 = \bar{w}\, z^{1/3} F^{-1/3}$

$\therefore$ Pi Th^m says: $\Pi_2 = f(\Pi_1)$

$\therefore$ $\bar{w}\, z^{1/3} F^{-1/3} = f\left(\frac{r}{R}\right)$

$$\therefore \quad \boxed{\bar{w} = \frac{F^{1/3}}{z^{1/3}} f\left(\frac{r}{R}\right)}$$

So along centerline $\bar{w}$ decreases with height. [In contrast, in the laminar plume $\bar{w}$ doesn't change with height along centerline]. So turbulence saps the strength of the turbulent plume.

To get buoyancy $g\left(\frac{\overline{T - T_0}}{T_0}\right)$:

list of parameters: $g\frac{(\overline{T - T_0})}{T_0}$, $\frac{r}{R}$, F, z
(so $m = 4$)

dimensions: $\frac{L}{T^2}$, 1, $\frac{L^4}{T^3}$, L

two dimensions, L and T $(k = 2)$

\# of pi groups is $m - k = 4 - 2 = 2$

$\Pi_1 = r/R$

$\Pi_2 = g\,\overline{\frac{(T-T_0)}{T_0}}\, z^a F^b$

$[\Pi_2] = \frac{L}{T^2} L^a \left(\frac{L^4}{T^3}\right)^b = L^{1+a+4b} T^{-2-3b}$

$\therefore 1+a+4b=0$ and $-2-3b=0 \rightarrow \boxed{b=-\frac{2}{3}}$

$\therefore 1+a+4\left(-\frac{2}{3}\right)=0 \rightarrow 1+a-\frac{8}{3}=0$ $\quad a=\frac{5}{3}$

$\therefore \Pi_2 = g\,\overline{\frac{(T-T_0)}{T_0}}\, z^{5/3} F^{-2/3}$

Pi Th^m says $\Pi_2 = f(\Pi_1)$

$\therefore g\,\overline{\frac{(T-T_0)}{T_0}}\, g\, z^{5/3} F^{-2/3} = f\left(\frac{r}{R}\right)$

$$\therefore \boxed{g\,\overline{\frac{(T-T_0)}{T_0}} = \frac{F^{2/3}}{z^{5/3}} f\left(\frac{r}{R}\right)}$$

So Temp perturbation decreases rapidly with z. Much more rapidly than laminar plume ($1/z$).

Some function. Not necessarily same as one for $\overline{w}$

Lastly, determine plume radius R.

List of parameters: R, F, z (m=3)

dimensions: L, $\frac{L^4}{T^3}$, L 2 dimensions; L and T $(k=2)$

$n = m - k = 3 - 2 = 1$ pi number

an obvious choice is: $\Pi_1 = \frac{R}{z}$ That's it. 1 pi group is all we need.

Note that the "leftover" parameter F didn't get used. No problem. F did get used in the determination of $\bar{w}$ and buoyancy so it's not like we've completely abandoned it. But it can't get used in analysis of R because there's no way the time dimension in F can cancel with dimensions in R or z. In other words: R cannot be a function of F.

So we really have: parameters R, z $(m=2)$
dimensions L, L (~~k=~~ $m=1$, $k=1$)

$2-1=1$ pi number

$\Pi_1 = R/z$

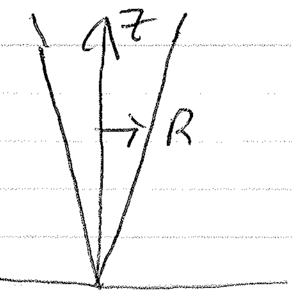
So Pi Th^m yields: $f(\Pi_1)=0$

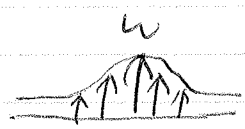
$\Pi_1 = f^{-1}(0) = \text{const}$

$\therefore R/z = \text{const}$

$\therefore \boxed{R = \overbrace{\alpha}^{\text{const}} z}$

Get a conical plume with apex at the source.





Experiment and theory (Rouse, Yih, & Humphreys, 1952; Yih 1951) show $\bar{w}$ is nearly 0 outside plume and nearly Gaussian within plume.

Expts and theory also show a lateral entrainment of fluid into the plume associated with an increase of mass flux with z.

Even though $\bar{w}\downarrow$ with z, $R\uparrow$ with z in such a manner that mass flux $\int_A \bar{w}\,dA \uparrow$ with z.

A ← horiz cross section through plume

This necessitates lateral inflow (entrainment) via mass conservation.

Now let calculate the entrainment velocity into the turbulent plume over a point source in an unstratified environment.

The relevant mass conservation eqn is the incompressibility condn (since we've been working under the Boussinesq approx), $\nabla\cdot\vec{v}=0$. Since flow is axisymmetric, $\nabla\cdot\vec{v}=0$ can be written as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}_r\right)+\frac{\partial\bar{w}}{\partial z}=0.$$

Mult by r and integrate from $r=0$ to $r=R$.

$$\int_0^R \frac{\partial}{\partial r}(r\bar{v}_r)\,dr = -\int_0^R r\frac{\partial\bar{w}}{\partial z}\,dr$$

Can immediately integrate l.h.s. (it's a perfect differential). On r.h.s., replace upper limit R by ∞; it's an approximation but a good one since $\bar{w}$ is nearly 0 outside of the plume.

$$\therefore \left[r v_r\right]\Big|_0^R = -\int_0^\infty r\frac{\partial\bar{w}}{\partial z}\,dr$$

$$\therefore R v_r(R,z) = -\frac{\partial}{\partial z}\int_0^\infty r\bar{w}\,dr$$

Note: if upper limit was still R, could not pull $\frac{\partial}{\partial z}$ in front of integral.

Now use $\bar{w}=\frac{\text{const}}{z^{1/3}}f\left(\frac{r}{R}\right)$

$$R\,U_r(R,z) = -\frac{\partial}{\partial z}\int_0^\infty \frac{r\,\text{const}\,f(\frac{r}{R})}{z^{1/3}}\,dr$$

$$= -\frac{\partial}{\partial z}\int_0^\infty \frac{R^2\,\text{const}}{z^{1/3}}\,\frac{r}{R}\,f\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)$$

$$= -\frac{\partial}{\partial z}\left[\frac{R^2}{z^{1/3}}\,\text{const}\underbrace{\int_0^\infty \frac{r}{R}\,f\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)}_{\text{just some number. Not a } f^n \text{ of } r \text{ or } R \text{ or } z.}\right]$$

now use $R = \alpha z$

so $R^2 = \alpha^2 z^2$

$$= -\frac{\partial}{\partial z}\left[\frac{R^2}{z^{1/3}}\times\text{another const}\right]$$

$$= -\frac{\partial}{\partial z}\left(z^{5/3}\times\text{nuther const}\right)$$

$$= \text{nuther nuther const}\times z^{2/3}$$

$$\therefore\ \overline{U}_r(R,z) = \text{nuther const}\,\frac{z^{2/3}}{R} = \frac{\text{const}}{z^{1/3}}$$

So entrainment velocity ↓ with z $\left(\overline{U}_r \propto \frac{1}{z^{1/3}}\right)$ but the area on which it acts (plume area is conical) ↑ with z.

We've shown that entrainment velocity ~~U_R~~ $\overline{U}_r(R,z)$ varies with height in same manner as the updraft speed along centerline $\overline{w}(0,z)$; both vary as $\frac{1}{z^{1/3}}$. How does $\overline{U}_r(R,z)$ compare to the horizontal average of $\overline{w}$ across the plume?

Horiz ave of $\overline{w}$ (ensemble ave) across the plume is:

horiz ave of ensemble ave

$$\overline{\overline{w}} \equiv \frac{\int_A \overline{w}\, dA}{\int_A dA} \qquad \left[\text{ave} = \frac{\text{integral}}{\text{interval}}\right]$$

where A extends across plume area

$$= \frac{2\pi \int_0^R \overline{w}\, r\, dr}{2\pi \int_0^R r\, dr} = \frac{2\int_0^R \overline{w}\, r\, dr}{R^2}$$

$r/R = 1$ when $r=R$

$$= \frac{2}{R^2} \int_0^1 R^2\, \overline{w}\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

use $\overline{w} = \frac{\text{const}\, f(r/R)}{z^{1/3}}$

$$= \frac{2\ \text{const}}{z^{1/3}} \underbrace{\int_0^1 f\left(\frac{r}{R}\right)\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)}_{\text{just some number}}$$

$$= \frac{\text{nother const}}{z^{1/3}}$$

So $\overline{U}_R(R,z) = \text{const}\ \overline{\overline{w}}$

entrainment velocity varies with z like centerline velocity and like mean updraft velocity (they all vary like $1/z^{1/3}$).

This result has been deduced (not assumed) in this theory for a turbulent plume in an unstratified environment. But this same relation will be assumed (not deduced) in theories of turbulent plumes in stratified environments.