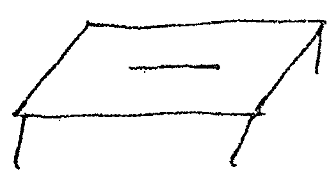
Hydrodynamic Stability

A flow that satisfies the eq^{ns} of motion may still not be physically realizable if small disturbances of it grow with time, i.e. if the flow is unstable.

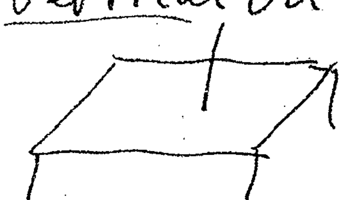
A non-fluid example: Consider stability of a needle on a table:

Case 1: Needle is horizontal on table.

Case 2: Needle is vertical on table.



stable



unstable

In both cases the "equation of motion" is the same: downward force ~~[illegible]~~ of gravity on needle is balanced by upward reactive force of table on needle, yielding 0 acceleration. But small vibrations ~~[illegible]~~ make vertical needle fall, while horiz needle hardly moves. Vertical needle orientation is unstable!

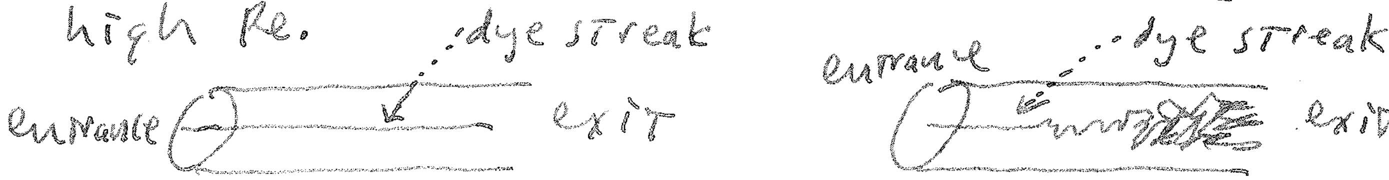
<u>Another non-fluid example</u>: Consider a marble in a basin.

stable to small and large disturbances

unstable

stable to small disturbances but unstable to large ones.

<u>A fluid example</u>: Consider flow of fluid in a tube. Laminar solution discovered by Poiseuille and Stokes, independently. Using dye to visualize the flow, Osborne Reynolds (1883) showed ~~that at~~ experimentally that at high flow speeds (well, high Reynolds numbers $Re \equiv UL/\nu$) the laminar solution broke down, and the flow became turbulent, i.e. the laminar solution was <u>unstable</u> at high Re.



low Re flow is <u>stable</u>

high Re flow is <u>unstable</u>

<u>Another fluid example</u>: Rayleigh-Bénard convection (thermal instability).

Consider fluid at rest sandwiched btw a lower plate and an upper plate. The lower plate is warm, and the upper plate is cool. If the temp difference btw the plates is suitably small, the fluid still remains at rest. Any motion would be opposed by friction. But if the temp difference is large enough can get overturning motion (convection) despite friction.

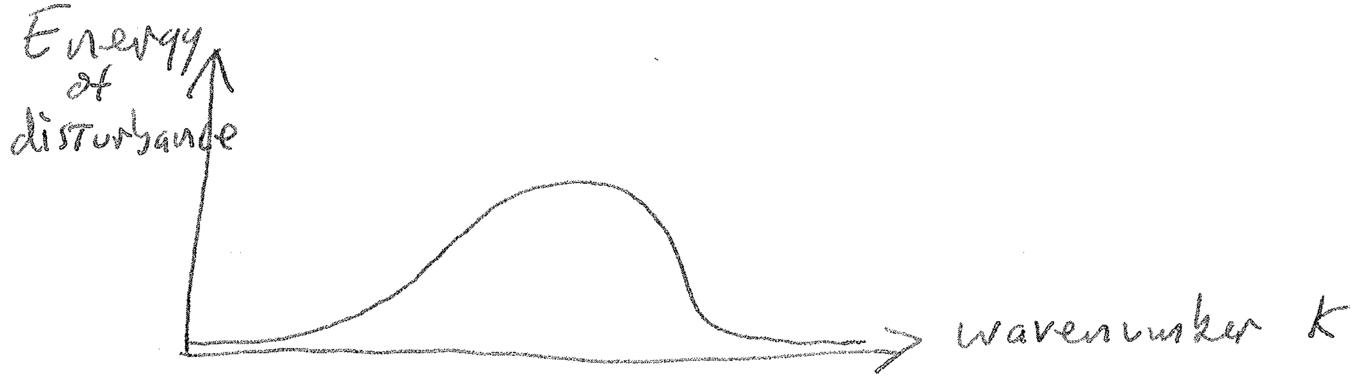
Hydrodynamic stability is concerned with characterizing the stability/instability of specific "base-state" fluid motions to specific types of disturbances. The flow being studied is usually pretty simple, but satisfies the nonlinear equations of motion exactly.

Linear stability theory: The disturbance is considered to be infinitesimal. The subsequent evolution of the flow is studied with linearized equations of motion. Theory is valid for all times for a stable case but only for short times for an unstable case (why?).

Nonlinear stability theory: the disturbance is not infinitesimal: it's of "finite amplitude." Must use nonlinear eq^{ns} of motion to study the evolution.

We will be examining linear stability of a few simple but important idealized flows.

Consider an idealized flow satisfying the nonlinear eq^{ns} of motion. Consider an infinitesimal disturbance of this flow. Disturbance might have an x-dependence alone, or x, y dependence or x, y, z dependence or maybe a Θ-dependence or r-dependence or r, Θ dependence or ... Disturbance will, in general, be characterized by a continuum of wavenumbers.



All wavenumbers can be present in general disturbances.

In linear stability theory can study stability of flow to arbitrary general initial disturbances by studying reaction to individual wavenumber disturbances.

A flow is stable if it's stable to all wavenumbers.

A flow is unstable if it's unstable to any wavenumber (even one). "One bad apple ruins it for everyone."

In a flow with diffusion, a flow is stable if the initial disturbance dies out ($\to 0$). Unstable if disturbance grows with time.

Rayleigh-Bénard Convection

[for more details see chapter on Instability in Kundu's "Fluid Mechanics", or esp Chandrasekhar's monograph "Hydrodynamic + Hydromagnetic Stability"]

Consider fluid layer sandwhich btw 2 horiz plates. Top plate is cold. Lower plate is hot. The plate temps are held fixed. [Note: for atmospheric flow a fixed heat flux condition $\partial T/\partial z$ = specified is more appropriate).

Neglect earth's rotation (see Chandrasekhar for treatment w/ Coriolis).

Initial state (pre-disturbance) has no motion and is horizontally homogeneous (thermally).

We'll work w/ Boussinesq eq^{ns} of motion.

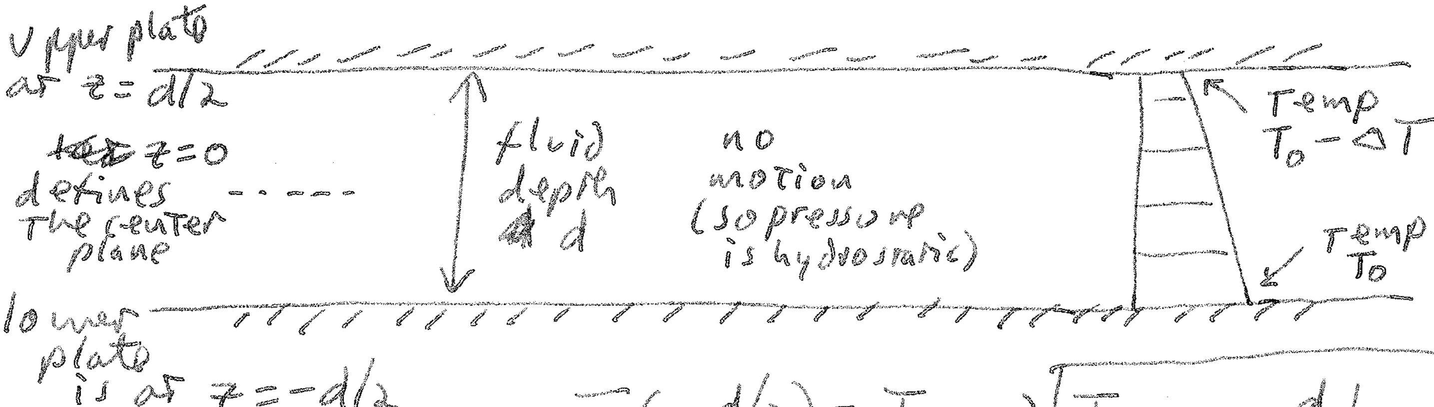
The thermodynamic energy eq^n for the initial state reduces to $\kappa \nabla^2 \bar{T} = 0$ ($\bar{T}$ is initial temp, and κ is thermal diffusivity)

well, really: $\dfrac{d^2\bar{T}}{dz^2} = 0$

integrate twice, get

$$\bar{T}(z) = a + bz$$

So initial temp varies linearly with z across the fluid layer. Can use bound. cond^{ns} to find a, b.



$\bar{T}(-d/2) = T_0 \rightarrow T_0 = a - \frac{d}{2} b$

$\bar{T}(d/2) = T_0 - \Delta T \rightarrow T_0 - \Delta T = a + \frac{d}{2} b$

2 eq^{ns} in 2 unknowns a, b

$\therefore$ find $b = -\dfrac{\Delta T}{d}$

$a = T_0 - \Delta T/2$

Note that ΔT is positive if $\bar{T}$ is to decrease with z.

So $\overline{T}(z) = T_0 - \frac{\Delta T}{d}\left(z + \frac{d}{2}\right)$

Temp gradient $\frac{d\overline{T}}{dz} = -\frac{\Delta T}{d}$ is const

Lapse rate $\Gamma \equiv -\frac{d\overline{T}}{dz} = \frac{\Delta T}{d} > 0$

$\therefore \quad \overline{T}(z) = T_0 - \Gamma\left(z + \frac{d}{2}\right)$

Consider disturbances of this initial state.

Boussinesq eq^{ns} of motion (written in terms of <u>full</u> pressure p and <u>full</u> density ρ):

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\rho}{\rho_0} g \delta_{i3} + \nu \nabla^2 v_i$$

where ρ_0 is a constant reference value of density.

Boussinesq form of thermodynamic energy:

$$\frac{\partial T}{\partial t} + v_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T$$

Mass conservation (incompressibility cond^n):

$$\frac{\partial v_i}{\partial x_i} = 0$$

Eq^n of state for a liquid (also a reasonable approx for a gas if temp changes are small):

$$\rho = \rho_0 \left[1 - \alpha (T - T_0)\right]$$

where T_0 is the value of temp where $\rho = \rho_0$, and

$\alpha \equiv -\frac{1}{\rho_0} \frac{d\rho}{dT}\Big|_{T=T_0}$ is coefficient of volume expansion. IT'S <u>POSITIVE</u>.

Initial state (pre-disturbance):

Thermo energy eq^n: $\kappa \nabla^2 \bar{T} = 0$

already found the sol^n: $\bar{T}(z) = T_0 - \Gamma(z + \frac{d}{2})$

where $\Gamma \equiv \frac{\Delta T}{d}$

Eq^n of motion: $0 = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x_i} - g[1 - \alpha(\bar{T} - T_0)]\delta_{i3}$

where $\bar{p}$ is initial state of pressure.

So $\frac{\partial \bar{p}}{\partial x} = 0$, $\frac{\partial \bar{p}}{\partial y} = 0$, $\frac{\partial \bar{p}}{\partial z} = -\rho_0 g[1 - \alpha(\bar{T} - T_0)]$

Introduce perturbation pressure p': $p = \bar{p}(z) + p'$

(so $p' = p - \bar{p}(z)$)

Similarly $T = \bar{T}(z) + T'$

$\therefore$ The eq^{ns} of motion for the disturbed flow are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x_i} - \frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} - g[1 - \alpha(\bar{T} + T' - T_0)]\delta_{i3} + \nu \nabla^2 u_i$$

Properties of $\bar{p}$ causes some cancellation, get

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} + g\alpha T' \delta_{i3} + \nu \nabla^2 u_i$$

Thermo Energy eq^n: $\frac{\partial T'}{\partial t} + u_j \frac{\partial T'}{\partial x_j} + u_j \frac{\partial \bar{T}}{\partial x_j} = \kappa \nabla^2 T'$

expand it

$$u_1 \underbrace{\frac{\partial \bar{T}}{\partial x_1}}_{\to 0} + u_2 \underbrace{\frac{\partial \bar{T}}{\partial x_2}}_{\to 0} + u_3 \frac{\partial \bar{T}}{\partial x_3} = w\frac{d\bar{T}}{dz} = -\Gamma w$$

Now linearize them:

(1) $$\frac{\partial v_i}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} + g\alpha T'\delta_{i3} + \nu\nabla^2 v_i$$

(2) $$\frac{\partial T'}{\partial t} - \Gamma w = \kappa \nabla^2 T'$$

also (3) $\dfrac{\partial v_i}{\partial x_i} = 0$ (The Boussinesq form of mass conservation is already linear).

This is really 5 eq^{ns} in 5 unknowns (eq^n (1) is really 3 eq^{ns} in one since $i = 1, 2, 3$).

Lets work on eliminating pressure pert. Take $\frac{\partial}{\partial x_i}$ of (1) and use (3), get:

$$0 = -\frac{1}{\rho_0}\underbrace{\frac{\partial^2 p'}{\partial x_i \partial x_i}}_{\to \nabla^2 p'} + g\alpha \frac{\partial T'}{\partial x_i}\delta_{i3} + 0$$

use substitution principle
$$\frac{\partial T'}{\partial x_i}\delta_{i3} = \frac{\partial T'}{\partial x_3} = \frac{\partial T'}{\partial z}$$

$$\therefore \nabla^2 p' = \rho_0 g\alpha \frac{\partial T'}{\partial z}$$

... and take Laplacian of vertical ($i=3$) comp of (1):

$$\frac{\partial}{\partial t}\nabla^2 w = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\nabla^2 p' + g\alpha\nabla^2 T' + \nu\nabla^4 w$$

we've shown it's $\rho_0 g\alpha \frac{\partial T'}{\partial z}$

$$\therefore \frac{\partial}{\partial t}\nabla^2 w = \underbrace{-g\alpha\frac{\partial^2 T'}{\partial z^2} + g\alpha\nabla^2 T'}_{\text{get some cancellation}} + \nu\nabla^4 w$$

$$\boxed{\frac{\partial}{\partial t}\nabla^2 w = g\alpha\nabla_H^2 T' + \nu\nabla^4 w}$$

where $\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is horizontal Laplacian

and thermo eq^n:

$$\boxed{\frac{\partial T'}{\partial t} - \Gamma w = \kappa\nabla^2 T'}$$

2 eq^{ns} in 2 unknowns, T', w

Now consider boundary cond^{ns}.

Impose impermeability cond^n on top and bottom boundaries ($\vec{v}\cdot\hat{n}=0$ where $\hat{n}$ is normal to solid bdry)

So: $w=0$ on $z=d/2$
$w=0$ on $z=-d/2$

Next consider 2 dif ~~sa~~ boundary scenarios: no-slip cond^n versus free-slip cond^n. Need to translate them into cond^{ns} for w.

No-slip cond^n: $u=v=0$ on $z=d/2$ and $z=-d/2$

Since u and v are 0 all along $z=d/2$ and $z=-d/2$, their horiz derivs are 0 $\therefore$ $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ on $z=\pm d/2$

$\therefore$ from ~~impermeabi~~ incompressibility condition,

$\frac{\partial w}{\partial z}=0$ on $z=d/2$ and $z=-d/2$

In contrast, for free-slip cond^n (stress-free cond^n),

$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=0$ on $z=\frac{d}{2}$ and $z=-\frac{d}{2}$

$\therefore \quad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) = 0$ and $\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z}\right)$ all along $z = \frac{d}{2}$ and $z = -\frac{d}{2}$

$$\therefore \quad \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$\therefore$, from incompressibility cond^n,

$$\underbrace{\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right.}_{0} + \left.\frac{\partial w}{\partial z}\right) = 0$$

$$\therefore \quad \frac{\partial^2 w}{\partial z^2} = 0$$

Lastly, for temperature, since the top and bottom temperatures are fixed at their initial hot/cold values, $T' = 0$ on $z = d/2$ and $z = -d/2$

So: For no-slip case:

$$w = \frac{\partial w}{\partial z} = T' = 0 \quad \text{for } z = \pm \frac{d}{2}$$

For free-slip case:

$$w = \frac{\partial^2 w}{\partial z^2} = T' = 0 \quad \text{for } z = \pm \frac{d}{2}$$

It is convenient to non-dimensionalize the independent variables.

$$\tilde{x} \equiv \frac{x}{d} \qquad \tilde{y} \equiv \frac{y}{d} \qquad \tilde{z} \equiv \frac{z}{d}$$

So $x = \tilde{x} d, \quad y = \tilde{y} d, \quad z = \tilde{z} d$

For non-dim time, note that $[\kappa] = \frac{L^2}{T}$, $[\nu] = \frac{L^2}{T}$ so can define a time scale as $\frac{d^2}{\kappa}$ or $\frac{d^2}{\nu}$. Make a choice and stick with it (doesn't matter which). OK. $\frac{d^2}{\kappa}$

$$\tilde{\tau} \equiv \frac{\tau}{d^2/\kappa} = \frac{\kappa \tau}{d^2} \quad \therefore \tau = \frac{d^2}{\kappa} \tilde{\tau}$$

So $\frac{\partial}{\partial x} = \frac{1}{d} \frac{\partial}{\partial \tilde{x}}, \quad \frac{\partial}{\partial y} = \frac{1}{d} \frac{\partial}{\partial \tilde{y}}$, etc

$$\nabla() = \frac{1}{d} \tilde{\nabla} \qquad \nabla^2() = \frac{1}{d^2} \tilde{\nabla}^2()$$

$$\frac{\partial}{\partial \tau} = \frac{\kappa}{d^2} \frac{\partial}{\partial \tilde{\tau}}$$

So
$$\frac{\partial}{\partial \tau} \nabla^2 w = g\alpha \nabla_H^2 T' + \nu \nabla^4 w$$

$$\frac{\partial}{\partial \tau} T' - \Gamma w = \kappa \nabla^2 T'$$

become

(mult by d^4/ν) $$\frac{\kappa}{d^2} \frac{\partial}{\partial \tilde{\tau}} \frac{1}{d^2} \tilde{\nabla}^2 w = \frac{g\alpha}{d^2} \tilde{\nabla}_H^2 T' + \frac{\nu}{d^4} \tilde{\nabla}^4 w$$

(mult by d^2/κ) $$\frac{\kappa}{d^2} \frac{\partial}{\partial \tilde{\tau}} T' - \Gamma w = \frac{\kappa}{d^2} \tilde{\nabla}^2 T'$$

eq^n of motion: $$\boxed{\left(\frac{1}{Pr}\frac{\partial}{\partial \tilde{t}} - \tilde{\nabla}^2\right)\tilde{\nabla}^2 w = \frac{g\alpha d^2}{\nu}\tilde{\nabla}_H^2 T'}$$

Thermo: $$\boxed{\left(\frac{\partial}{\partial \tilde{t}} - \tilde{\nabla}^2\right) T' = \frac{\Gamma d^2}{\kappa} w}$$

where $Pr \equiv \dfrac{\nu}{\kappa}$ is Prandtl number

The b.c. become:

$$w = 0 \quad \text{on} \quad \tilde{z} = \pm\frac{1}{2}$$

$$T' = 0 \quad \text{on} \quad \tilde{z} = \pm\frac{1}{2}$$

and either no slip $\dfrac{\partial w}{\partial \tilde{z}} = 0$ on $\tilde{z} = \pm\dfrac{1}{2}$

or stress-free (free-slip) $\dfrac{\partial^2 w}{\partial \tilde{z}^2} = 0$ on $\tilde{z} = \pm\dfrac{1}{2}$

Since the p.d.e.s are linear and homogeneous and the coefficients are constant, anticipate that we can find solutions of the form:

$$w = \hat{w}(\tilde{z})\, e^{i(k\tilde{x} + l\tilde{y}) + \sigma\tilde{t}}$$

$$T' = \hat{T}(\tilde{z})\, e^{i(k\tilde{x} + l\tilde{y}) + \sigma\tilde{t}}$$

where k and l are non-dimensional wavenumbers. Note that k and l must be real. If not, sol^n would blow up as $\tilde{x}$ (or $\tilde{y}$) goes to $+\infty$ or $-\infty$. So sol^n is allowed to oscillate in $\tilde{x}, \tilde{y}$ but not grow in $\tilde{x}, \tilde{y}$.

The growth rate σ can be complex:

$\sigma = \sigma_r + i\sigma_i$ where σ_r and σ_i are real.

Note that $e^{\sigma\tilde{\tau}} = e^{(\sigma_r + i\sigma_i)\tilde{\tau}} = \underbrace{e^{\sigma_r\tilde{\tau}}}_{\text{grows or decays in time}} \underbrace{e^{i\sigma_i\tilde{\tau}}}_{\text{oscillates in time}}$

If $\sigma_r > 0$ then $e^{\sigma_r\tilde{\tau}} \longrightarrow \infty$ as $\tau \to \infty$.

If $\sigma_r < 0$ then $e^{\sigma_r\tilde{\tau}} \longrightarrow 0$ as $\tau \to \infty$.

So, if $\sigma_r > 0$ for <u>any</u> k, l, the flow is <u>unstable</u>.

but if $\sigma_r < 0$ for <u>all</u> k, l, the flow is <u>stable</u>.

Plugging the normal mode expressions into the eq^n of motion, we get:

$$\left[\frac{\sigma}{Pr} - \left((ik)^2 + (il)^2 + \frac{d^2}{d\tilde{z}^2}\right)\right]\left((ik)^2 + (il)^2 + \frac{d^2}{d\tilde{z}^2}\right)\hat{w}$$

$$= \frac{g\alpha d^2}{\nu}\left((ik)^2 + (il)^2\right)\hat{T}$$

let $K^2 = k^2 + l^2$

$D^2 \equiv \frac{d^2}{d\tilde{z}^2}$ so eq^n of motion becomes

$$\boxed{\left[\frac{\sigma}{Pr} - \left(D^2 - K^2\right)\right]\left(D^2 - K^2\right)\hat{w} = -\frac{g\alpha d^2 K^2}{\nu}\hat{T}}$$

Similarly, the thermo eq^n becomes:

$$\boxed{\left[\sigma - (D^2 - K^2)\right] \hat{T} = \frac{\Gamma d^2}{\kappa} \hat{w}}$$

Further simplify by defining $W \equiv \frac{\Gamma d^2}{\kappa} \hat{w}$

so thermo eq^n becomes

(4) $$\boxed{\left[\sigma - (D^2 - K^2)\right] \hat{T} = W}$$

and $\frac{\Gamma d^2}{\kappa}$ times eq^n of motion becomes

$$\left[\frac{\sigma}{Pr} - (D^2 - K^2)\right](D^2 - K^2) W = -\frac{\Gamma g \alpha d^4}{\nu \kappa} K^2 \hat{T}$$

$Ra \equiv \frac{\Gamma g \alpha d^4}{\nu \kappa}$ is the Rayleigh number.

It's a ratio of de-stabilizing effect of adverse temperature gradient to stabilizing effect of viscosity and thermal diffusivity.

So:

(5) $$\boxed{\left[\frac{\sigma}{Pr} - (D^2 - K^2)\right](D^2 - K^2) W = -Ra K^2 \hat{T}}$$