1 Handout: cell patterns from pgs 43-52 of Chandrasekhar.

The <u>marginal state</u> separating stability from instability is given by $\sigma_r = 0$. However, as we will see, in the Rayleigh-Bénard scenario, $\sigma_i = 0$. $\therefore$ The marginal state in the R. B. problem is governed by $\underline{\sigma = 0}$.

So, let's prove that $\sigma_i = 0$ [a result known as the "principle of exchange of stability" for reasons that aren't clear]

To prove $\sigma_i = 0$ we want $\sigma$ to be in eq$^{ns}$ with terms that are clearly identifiable as <u>real</u> or <u>imaginary</u>. Note that $\hat{T}$ is complex but $\hat{T}$ times its complex conjugate $\hat{T}^*$ is <u>real</u> and <u>positive</u>. [for any complex # $p$: $pp^* = |p|^2$]

complex conjugate

Multiply (4) by $\hat{T}^*$ and integrate from bottom to top (i.e. from $\tilde{z} = -1/2$ to $\tilde{z} = 1/2$).

$$\therefore \int \hat{T}^* [\sigma - (D^2 - K^2)] \hat{T} \, d\tilde{z} = \int W \hat{T}^* d\tilde{z}$$

$$\sigma \int \hat{T}^* \hat{T} \, d\tilde{z} + K^2 \int \hat{T}^* \hat{T} \, d\tilde{z} - \boxed{\int \hat{T}^* D^2 \hat{T} \, d\tilde{z}} = \int W \hat{T}^* d\tilde{z}$$

int by parts

$$\int \hat{T}^* D^2 \hat{T} \, d\tilde{z} = \int D[\hat{T}^* D\hat{T}] \, d\tilde{z} - \int D\hat{T}^* D\hat{T} \, d\tilde{z}$$

$$= \underbrace{\left[\hat{T}^* D\hat{T}\right]_{-1/2}^{1/2}}_{\to 0 \text{ from b.c. } (\hat{T} = 0 \text{ at } \tilde{z} = \pm 1/2 \text{ so } \hat{T}^* = 0 \text{ '' '' ''})} - \int D\hat{T}^* D\hat{T} \, dz$$

$$\therefore \sigma \int \hat{T}^* \hat{T} \, d\tilde{z} + K^2 \int \hat{T}^* \hat{T} \, d\tilde{z} + \int D\hat{T}^* D\hat{T} \, d\tilde{z} = \int W \hat{T}^* d\tilde{z}$$

$$\therefore \quad \sigma \underbrace{\int |\hat{T}|^2 d\tilde{z}}_{I_1} + \underbrace{K^2 \int |\hat{T}|^2 d\tilde{z} + \int |D\hat{T}|^2 d\tilde{z}}_{I_2} = \int W \hat{T}^* d\tilde{z}$$

(A) $$\boxed{\sigma I_1 + I_2 = \int W \hat{T}^* d\tilde{z}}$$

where $I_1$ and $I_2$ are <u>positive</u>.

Now mult (5) by $W^*$ and int btw bottom/top.

$$\frac{\sigma}{Pr} \int W^* (D^2 - K^2) W d\tilde{z} - \int W^* (D^2 - K^2)^2 W d\tilde{z}$$
$$= -RaK^2 \int W^* \hat{T} d\tilde{z}$$

since $(D^2 - K^2)^2 = D^4 - 2K^2 D^2 + K^4$, get

$$\frac{\sigma}{Pr}\left[ -K^2 \int W^* W d\tilde{z} + \int W^* D^2 W d\tilde{z} \right]$$

now int. by parts →

$$- \int W^* D^4 W d\tilde{z} + 2K^2 \int W^* D^2 W d\tilde{z} - K^4 \int W^* W d\tilde{z}$$
$$= -Ra K^2 \int W^* \hat{T} d\tilde{z}$$

$$\int W^* D^2 W d\tilde{z} = \underbrace{\left[ W^* D W \right]_{-1/2}^{1/2}}_{\to 0 \text{ from b.c.}} - \int DW^* DW d\tilde{z}$$

→ 0 from b.c. [$W = 0$ at $\tilde{z} = \pm 1/2$
so $W^* = 0$ " " " "]

$$\int W^* D^4 W d\tilde{z} = \underbrace{\left[ W^* D^3 W \right]_{-1/2}^{1/2}}_{\to 0 \text{ from b.c.}} - \int DW^* D^3 W d\tilde{z}$$

$$= -\left[ \int D[DW^* D^2 W] d\tilde{z} - \int D^2 W^* D^2 W d\tilde{z} \right]$$

$$= -\underbrace{\left[ DW^* D^2 W \right]_{-1/2}^{1/2}}_{\text{it's } 0!} + \int D^2 W^* D^2 W d\tilde{z}$$

0 from b.c. because either $DW^* = 0$ (no slip) or $D^2 W = 0$ (shear free)

$$\therefore \frac{\sigma}{Pr}\left[-K^2\int W^* W d\tilde{z} - \int DW^* DW d\tilde{z}\right]$$

$$-\int D^2W^* D^2W d\tilde{z} - 2K^2\int DW^* DW d\tilde{z}$$

$$-K^4\int W^* W d\tilde{z} = -Ra K^2 \int W^* \hat{T} d\tilde{z}$$

mult by $-1$ and use $P^* P = |P|^2$

$$\therefore \frac{\sigma}{Pr}\underbrace{\left[K^2\int |W|^2 d\tilde{z} + \int |DW|^2 d\tilde{z}\right]}_{J_1}$$

$$+\underbrace{\int |D^2W|^2 d\tilde{z} + 2K^2\int |DW|^2 d\tilde{z} + K^4\int |W|^2 d\tilde{z}}_{J_2}$$

$$= Ra K^2 \int W^* \hat{T} d\tilde{z}$$

$$\therefore \ (B) \quad \boxed{\frac{\sigma}{Pr} J_1 + J_2 = Ra K^2 \int W^* \hat{T} d\tilde{z}}$$

where $J_1$ and $J_2$ are positive.

The complex conjugate of (A) is:

$$\sigma^* I_1 + I_2 = \int W^* \hat{T} d\tilde{z}$$ Use this in (B)!

$$\therefore \ (C) \quad \frac{\sigma}{Pr} J_1 + J_2 = Ra K^2 (\sigma^* I_1 + I_2)$$

Taking the real part of (C) yields

note:
$\sigma = \sigma_r + i\sigma_i$
so
$\sigma^* = \sigma_r - i\sigma_i$

$$\frac{\sigma_r}{Pr} J_1 + J_2 = Ra K^2 (\sigma_r I_1 + I_2)$$

$$\therefore \ \sigma_r \left(\frac{J_1}{Pr} - Ra K^2 I_1\right) + J_2 - Ra K^2 I_2 = 0$$

So, if $Ra < 0$ then $\sigma_r$ (pos) + pos = 0

$\therefore$ $\sigma_r$ must be <u>negative</u>

$\therefore$ <u>stability</u>

note that $Ra < 0$ corresponds to stably-stratified case ($\Gamma < 0$ i.e. $\frac{dT}{dz} > 0$).

If $Ra > 0$ (Rayleigh-Bénard scenario; Topheavy case) then taking real part of (C) is inconclusive. Need more info. Okay, so now consider the <u>imaginary</u> part of (C):

$$\frac{\sigma_i}{Pr} J_1 = -Ra K^2 I_1 \sigma_i$$

$$\sigma_i \left( \frac{J_1}{Pr} + Ra K^2 I_1 \right) = 0$$

So if $Ra > 0$ then $\sigma_i$ (pos) = 0

$\therefore$ <u>Must have $\sigma_i = 0$</u>!

So $\sigma$ is <u>real</u> for Rayleigh-Bénard prob.

So in Rayleigh-Bénard prob. the <u>marginal state</u> ($\sigma_r = 0$) is actually associated with $\sigma = 0$.

So, eq$^{ns}$ (4) and (5) in the marginal state become:

$$(4') \quad (D^2 - K^2)\hat{T} = -W$$

$$(5') \quad (D^2 - K^2)^2 W = Ra K^2 \hat{T}$$

Eliminate $\hat{T}$ by taking $(D^2-K^2)$ of (5'):

$$\boxed{(D^2-K^2)^3 W = -Ra\,K^2 W}$$

$6^{th}$ order linear homog. const coeff. ode.

b.c.: $W = 0$
$\hat{T} = 0$, which from (5') means $(D^2-K^2)^2 W = 0$
and either $DW = 0$ (no slip)
or $D^2 W = 0$ (stress free)
all on $\tilde{z} = \pm\frac{1}{2}$

Note that the trivial sol$^n$ $W(\tilde{z}) = 0$ everywhere satisfies the o.d.e. and all b.c. If any non-trivial solution exists we expect it will only be for a set of special values of $Ra\,K^2$. This is an eigenvalue problem.

The easiest scenario (but not most realistic) is the ~~stress~~ slip free case. Let's examine that case.

So $\boxed{W=0}$ and $\boxed{D^2W=0}$ on $\tilde{z} = \pm\frac{1}{2}$

and since $(D^2-K^2)^2 W = 0 \rightarrow D^4 W - 2K^2 \underbrace{D^2 W}_{0} + K^4 \underbrace{W}_{0} = 0$

$\therefore$ $\boxed{D^4 W = 0}$ on $\tilde{z} = \pm\frac{1}{2}$

Seek sol$^n$ of the form $W = A \sin[n\pi(\tilde{z} - \frac{1}{2})]$

easy to show that it satisfies the b.c.:

$W(\frac{1}{2}) = A\sin(n\pi 0) = A\sin 0 = 0$ ✓
$W(-\frac{1}{2}) = A\sin(-n\pi) = -A\sin n\pi = 0$ ✓

Can show $D^2 W = 0$ on $\tilde{z} = \pm\frac{1}{2}$
[show it]

Plug $W = A\sin[n\pi(\tilde{z} - \frac{1}{2})]$ into the o.d.e.

Use fact that $DW = n\pi A\cos[n\pi(\tilde{z} - \frac{1}{2})]$

$$D^2 W = -n^2\pi^2 A\sin[n\pi(\tilde{z} - \tfrac{1}{2})]$$

$\therefore$ ode becomes:

$$(-n^2\pi^2 - K^2)^3 = -Ra\,K^2$$

$$\therefore \quad Ra = \frac{(n^2\pi^2 + K^2)^3}{K^2}$$ an eigenvalue relation

The critical Rayleigh number is the lowest value of Ra that yields a marginal state. This is obviously associated with $n=1$ (hence $n=1$ is the gravest mode) and a particular $K \to$ the $K$ that minimizes Ra.

So critical Ra satisfies:

$$\left(\frac{d}{dK} Ra\right)\Big|_{K=K_c} = 0$$

$$\left[\frac{d}{dK} \frac{(\pi^2 + K^2)^3}{K^2}\right]_{K=K_c} = 0$$

$$\frac{3(\pi^2+K_c^2)^2\, 2K_c}{K_c^2} - \frac{2}{K_c^3}(\pi^2+K_c^2)^3 = 0$$

$$\frac{3}{K_c} - \frac{(\pi^2+K_c^2)}{K_c^3} = 0 \rightarrow 3K_c^2 - \pi^2 - K_c^2 = 0$$

$$K_c^2 = \frac{\pi^2}{2}$$

$\therefore \boxed{K_c = \frac{\pi}{\sqrt{2}}}$ critical wavenumber

$\therefore$ critical wavelength is

$$\lambda_c = \frac{2\pi}{K_c} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$$

So critical dimensional wavelength is $2\sqrt{2}\,d$

Critical Rayleigh number for this ~~stр~~ stress free case is:

$$Ra_c = \frac{\left(\pi^2 + \frac{\pi^2}{2}\right)^3}{\pi^2/2} = \frac{\left(\frac{3^3}{2^3}\right)\pi^6}{\pi^2/2} = \frac{27}{4}\pi^4$$

$$\simeq 657$$

So, for $Ra < 657$ the flow is stable to disturbances of any wavelength.

But for Ra just exceeding 657, there is one wavelength for which the flow is unstable ($\lambda_c = 2\sqrt{2}\,d$). This will be the observed wavelength of the thermal instability.

For the case of rigid/rigid boundaries, the analysis is much more difficult (see Kundu, or Chandrasekhar or Drazin + Reid). Here

$Ra_c = 1101$ and $K_{cr} = 2.68$

<u>Cell Patterns.</u> Lab Experiments reveal that at the onset of instability, disturbances grow exponentially at first but then settle down to a steady-state <u>nonlinear</u> flow.

Our linear stability analysis yielded the horiz wavenumber of the thermal instability but not the pattern ($K^2 = k^2 + l^2$; we know $K$ but not $k$ or $l$).

If experimental conditions introduce no preferred horiz$^l$ dir$^{ns}$ (i.e. horiz$^l$ isotropy) then convective cells should be <u>regular polygons</u>: equilateral triangles, squares and hexagons.

Bénard observed hexagons in his lab exp$^{ts}$ but his flows were probably not closely related to ours (because of surface tension).

Hexagons observed in atmospheric mesoscale cellular convective flows.

Can get hexagons in <u>very</u> hot cup of coffee (gently add cream at side of cup → it's good for visualization)

If Ra ↑ above critical values, get coalescence of cells, get rolls.

Turbulence when Ra > 5 × 10$^4$

→ Examine handout on <u>cell patterns</u>.

Linear theory cannot predict cell shape $k, l$ or sense of flow (up or down in cell. Obs suggest rising motion in center of cell in a liquid, but sinking in a gas. But dependence of ν on Temp may be important here (cannot be studied w/ Boussinesq model).