Centrifugal Instability

Consider stability of axisymmetric vortices.

Linear theory is well-developed for inviscid and viscous Couette flow and for inviscid flows with arbitrary angular momentum profile (see Kundu, Chandrasekhar, Drazin + Reid).

We're just going to look at inviscid case.

Rayleigh's Inviscid Criterion

Consider an axisymmetric, inviscid, steady-state vortex w/ general tangential velocity profile $v = V(r)$, and radial velocity $u = 0$ and vert velocity $w = 0$. The only non-trivial eq^n of motion is radial comp:

$$\underbrace{-\frac{V^2}{r}}_{\text{centripetal acceleration}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial r}}_{\text{radial component of the pressure gradient force (p.g.f.)}}$$

Can integrate it to get p-field consistent with any tangential velocity profile $V(r)$.

Now consider axisymmetric disturbances of this vortex (can consider more general 3D disturbances but lab exp^ts ~~show~~ of Couette flow show that axisymmetric disturbances are first to appear).

Consider an idealized disturbance such that fluid rings at two radii are interchanged. Do the rings want to return to their starting configuration (stable situation) or do they continue moving away from each other?

Consider inviscid axisymmetric eq^{ns} of motion in cylindrical coordinates:

radial eq^n of motion (1) $$\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

Tangential eq^n of motion (2) $$\frac{Dv}{Dt} + \frac{uv}{r} = 0$$

vertical eq^n of motion (3) $$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$ ← perturbation pressure (deviation from hydrostatic pressure)

u = radial velocity component.
v = Tangential " "
w = vertical " "

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z} \qquad \left(\frac{1}{r}\frac{\partial()}{\partial\theta} \text{ is } 0 \text{ since axisym.}\right)$$

incompressibility condition (4) $$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

<u>Initial vortex</u> specified by: $u = w = 0$, $v = V(r)$ arbitrary profile

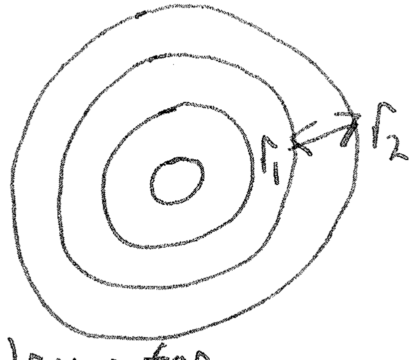
Can rewrite (2) as: $$r\frac{Dv}{Dt} + uv = 0$$

$$r\frac{Dv}{Dt} + v\frac{Dr}{Dt} = 0$$

$$\therefore \frac{D}{Dt}(rv) = 0$$

So angular momentum $\Gamma \equiv rv$ is <u>conserved</u> for a fluid parcel (or axisymmetric ring of fluid in an axisymmetric flow).

Assume that rv is conserved even during the <u>switcheroo</u> (interchange of fluid rings at radii r_1 and r_2).



drawn for $r_2 > r_1$

So fluid originally at r_1 has angular momentum $r_1 V(r_1)$, and after it has been displaced to radius r_2 it still has ang momentum $\overline{r_1 V(r_1)}$.

So, after the switcheroo:

$$\therefore \quad r_2 \underset{\substack{\uparrow \\ \text{note, small } v}}{v}(r_2) = r_1 \underset{\substack{\uparrow \\ \text{note; big } V}}{V}(r_1)$$

$$\therefore \quad v(r_2) = \frac{r_1}{r_2} V(r_1)$$

Assuming the p.g.f. is not affected by the switcheroo (which involves interchange of only an infinitesimal amount of fluid), the p.g.f. at r_2 is still given by:

$$-\frac{1}{\rho}\left.\frac{\partial p}{\partial r}\right|_{r_2} = -\frac{V^2(r_2)}{r_2}$$

So, immediately after the switcheroo, (1) at outer radius $r' = r_2$ becomes:

use above expression for $v(r_2)$

$$\left.\frac{Du}{Dt}\right|_{r_2} - \frac{v_2^2}{r_2} = -\frac{1}{\rho}\left.\frac{\partial p}{\partial r}\right|_{r_2}$$

$$\therefore \quad \left.\frac{Du}{Dt}\right|_{r_2} - \frac{r_1^2}{r_2^3} V^2(r_1) = \frac{-V^2(r_2)}{r_2}$$

$$\therefore \quad \left.\frac{Du}{Dt}\right|_{r_2} = \frac{1}{r_2^3}\left(r_1^2 V^2(r_1) - r_2^2 V^2(r_2)\right)$$

$$= \frac{1}{r_2^3}\left[\Gamma^2(r_1) - \Gamma^2(r_2)\right]$$

So, if squared angular momentum Γ^2 of initial vortex increases with radius then $\Gamma^2(r_2) > \Gamma^2(r_1)$

and so $\left.\frac{Du}{Dt}\right|_{r_2} < 0$ and so parcel tries to

return to its pre-switcheroo radius. This is the stable case.

But if Γ^2 ~~decreases~~ decreases w/radius then

$\Gamma^2(r_2) < \Gamma^2(r_1)$ and so $\left.\frac{Du}{Dt}\right|_{r_2} > 0$ and parcel

accelerates radially outward, away from its starting position. This is the unstable case.

So we get Rayleigh's criterion for an inviscid vortex:

An axisymmetric inviscid vortex is unstable to axisymmetric disturbances if $\frac{d\Gamma^2}{dr} < 0$ and is stable if $\frac{d\Gamma^2}{dr} > 0$.

e.g. solid body vortex:

$V(r) = \Omega r$ where Ω is const

$\therefore\ \Gamma = \Omega r^2$

so $\Gamma^2 = \Omega^2 r^4$

$\frac{d\Gamma^2}{dr} = 4\Omega^2 r^3 > 0$

$\therefore$ stable

e.g. vr-vortex (potential vortex):

$$V = \frac{c}{r} \quad \text{where } c \text{ is const}$$

$$\therefore \Gamma = c$$

$$\therefore \Gamma^2 = c^2$$

$$\therefore \frac{d\Gamma^2}{dr} = 0 \qquad \underline{\text{neutrally stable}}$$

e.g. $$V = \frac{c}{r^2}$$

$$\therefore \Gamma = \frac{c}{r}$$

$$\therefore \Gamma^2 = \frac{c^2}{r^2}$$

$$\therefore \frac{d\Gamma^2}{dr} = -\frac{2c^2}{r^3} < 0 \quad \therefore \underline{\text{unstable}}.$$

A less hand-wavy derivation of Rayleigh's criterion: linearize (1)–(4) about the base-state (initial) vortex, $u=0$, $w=0$, $v=V(r)$ and $-\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{V^2}{r}$

$$(1') \quad \frac{\partial u}{\partial t} - 2\frac{Vv}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r}$$

p' is pert pressure

$$(2') \quad \frac{\partial v}{\partial t} + \frac{uV}{r} + u\frac{dV}{dr} = 0$$

$$(3') \quad \frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$$

$$(4') \quad \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

These u, v, w, p' are all intended to be perturbations from the initial state variables

To eliminate v, take $\frac{\partial}{\partial \tau}$ (1′) and use (2′):

$$(\star)\quad \frac{\partial^2 u}{\partial \tau^2} - 2\frac{V}{r}\left(-u\frac{V}{r} - u\frac{dV}{dr}\right) = -\frac{1}{\rho}\frac{\partial}{\partial \tau}\frac{\partial p'}{\partial r}$$

To eliminate w, take $\frac{\partial}{\partial z}$ (3′) and use (4′):

$$\underbrace{\frac{\partial}{\partial z}\frac{\partial w}{\partial \tau}}_{\text{interchange order of differentiation}} = -\frac{1}{\rho}\frac{\partial^2 p'}{\partial z^2}$$

$$\frac{\partial}{\partial \tau}\left(\frac{\partial w}{\partial z}\right) \xrightarrow[\text{(4′)}]{\text{use}} \frac{\partial}{\partial \tau}\left(-\frac{\partial u}{\partial r} - \frac{u}{r}\right)$$

$$\therefore \quad (\star\star)\quad \frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = \frac{1}{\rho}\frac{\partial^2 p'}{\partial z^2}$$

To eliminate p', take $\frac{\partial^2}{\partial z^2}$ ($\star$) and use ($\star\star$):

$$\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2 u}{\partial \tau^2} + 2\frac{V}{r}u\left(\frac{dV}{dr} + \frac{V}{r}\right)\right] = -\frac{1}{\rho}\frac{\partial}{\partial \tau}\frac{\partial}{\partial r}\frac{\partial^2 p'}{\partial z^2}$$

$$= -\frac{\partial}{\partial \tau}\frac{\partial}{\partial r}\frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)$$

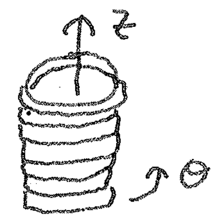
$$\therefore \quad \frac{\partial^2}{\partial \tau^2}\left[\frac{\partial^2 u}{\partial z^2} + \underbrace{\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)}_{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}}\right] + \underbrace{2\frac{V}{r}\left(\frac{dV}{dr} + \frac{V}{r}\right)}_{2\frac{V}{r}\left(\frac{1}{r}\frac{d}{dr}(rV)\right)}\frac{\partial^2 u}{\partial z^2} = 0$$

$$2\frac{V}{r}\left(\frac{1}{r}\frac{d}{dr}(rV)\right) \rightarrow \underbrace{\frac{2(rV)}{r^3}\frac{d}{dr}(rV)}_{\frac{1}{r^3}\frac{d}{dr}(rV)^2}$$

$$\therefore \boxed{\frac{\partial^2}{\partial \tau^2}\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + \Phi \frac{\partial^2 u}{\partial z^2} = 0}$$

where $\Phi \equiv \frac{1}{r^3}\frac{d}{dr}(rV)^2$ is the <u>Rayleigh discriminant</u>.

Lab experiments (vortex btw 2 concentric rotating cylinders) show instability sets in as an axisymmetric disturbance that's "wavy" in the axial dir^n. Imagine vertical stack of doughnuts:



$$\frac{\partial}{\partial \theta}(\,) = 0$$

So lets work with disturbances of the form:

$$u = \hat{u}(r)\, e^{ikz + \sigma\tau}$$

plug into the p.d.e., get the o.d.e.:

$$\sigma^2\left(\frac{d^2\hat{u}}{dr^2} + \frac{1}{r}\frac{d\hat{u}}{dr} - \frac{\hat{u}}{r^2} - k^2\hat{u}\right) - k^2\Phi u = 0$$

b.c.: impermeability cond^n on inner and outer cylinder: $\hat{u} = 0$ on cylinders.

-handouts on K.H. billows
-handout on prob below

Rayleigh's criterion for an inviscid vortex (cont^d)

Last class we showed radial velocity comp u in a perturbed (axi-symmetrically) vortex satisfies the p.d.e.:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+\Phi\frac{\partial^2 u}{\partial z^2}=0$$

where $\Phi \equiv \frac{1}{r^3}\frac{d}{dr}(rV)^2$ is the Rayleigh discriminant.

Consider trial sol^n of the form:

$$u=\hat{u}(r)\,e^{ikz+\sigma t}$$

Plug it into the p.d.e, get the o.d.e.:

$$\boxed{\frac{d^2\hat{u}}{dr^2}+\frac{1}{r}\frac{d\hat{u}}{dr}-\frac{\hat{u}}{r^2}-k^2\hat{u}-\frac{k^2}{\sigma^2}\Phi\,\hat{u}=0}$$

Rewrite o.d.e. in terms of a new independent variable:

$$s\equiv\frac{r^2}{2}$$

and new dependent variable:

$$\chi\equiv r\hat{u}$$

Get:

$$\boxed{\frac{d^2\chi}{ds^2}=\frac{k^2}{2s}\left(1+\frac{\Phi}{\sigma^2}\right)\chi}$$

simpler than above ode. Note there's no first deriv term in here.

with b.c.: $\chi=0$ on inner and outer cylinders (or on $r=0$ instead of inner cyl.)

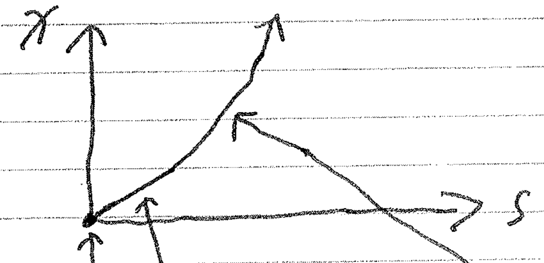
~~Exer~~ Problem: Derive this eq^n for χ. [give handout]

In order to meet the b.c. ($\chi = 0$ at two radii) must have:

$$\frac{d^2\chi}{ds^2} = (\text{something negative}) \text{ times } \chi$$

Why? Interpret the o.d.e. graphically. First lets suppose we have $\frac{d^2\chi}{ds^2} = (\text{positive})\,\chi$. So curvature has same sign as χ itself. Start at $s=0$ where $\chi=0$. Case (i) χ starts out with a positive slope. Case (ii) χ starts out with a negative slope.

case (i)

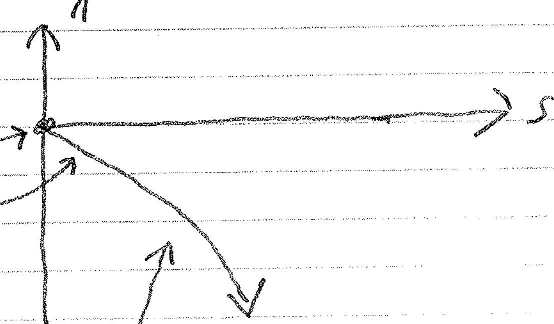


$\chi=0$

starts with a positive slope so χ becomes positive

since $\chi > 0$, the curvature is positive, and χ peels away from s axis. No way to satisfy the second b.c. (can't get χ to return to 0)

case (ii)



$\chi=0$

starts out with a negative slope so χ becomes negative

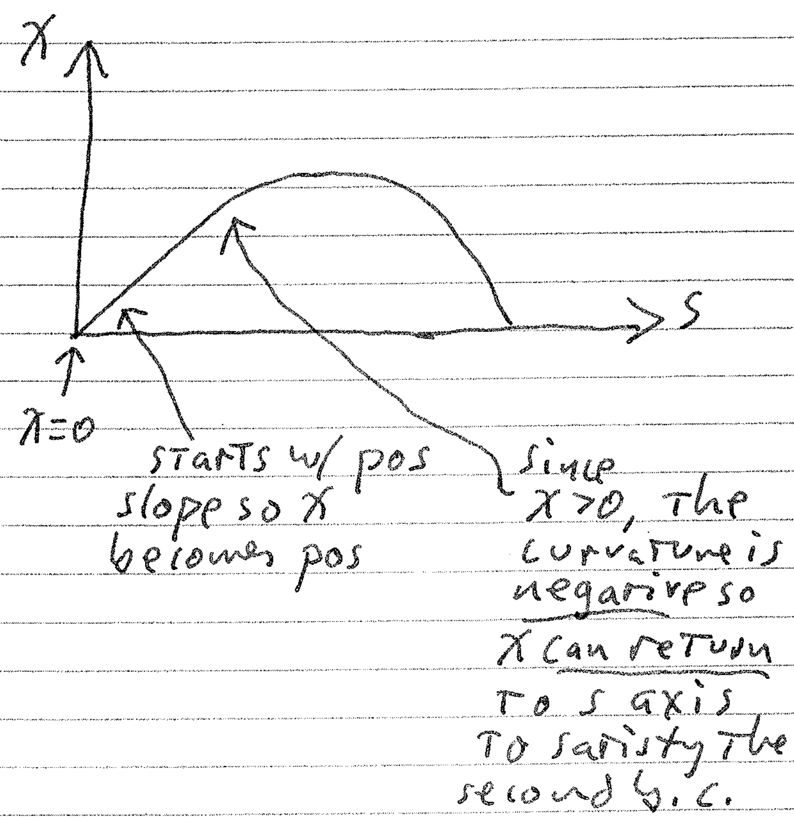
since $\chi < 0$, the curvature is negative and so χ still peels away from the s axis. No way for χ to return to 0.

So, if $\frac{d^2\chi}{ds^2}$ = (positive) χ, impossible to satisfy the b.c.. Dead in the water.

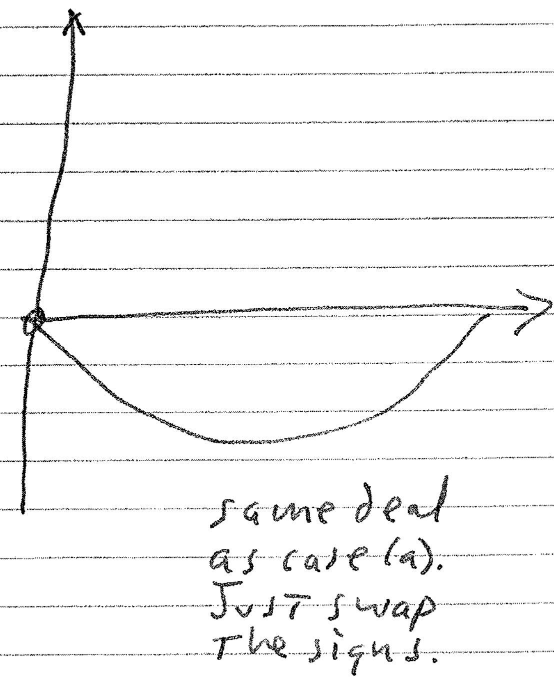
So now lets see about $\frac{d^2\chi}{ds^2}$ = (negative) χ.

In this case, the curvature has <u>opposite</u> sign of χ. Consider two cases (as before): Case (a) χ starts w/ pos slope. Case (b) χ starts w/ neg slope.

case (a)



case (b)



So, the only way to have a non-trivial (i.e. dif from $\hat{U}=0$ everywhere) sol^n is to have:

$$\frac{d^2\chi}{ds^2} = (\text{negative})\,\chi$$

$$\therefore \quad \frac{k^2}{2S}\left(1+\frac{\Phi}{\sigma^2}\right) < 0$$

Since $\frac{K^2}{2S} = \frac{k^2}{r^2} > 0$, must have $\boxed{1 + \frac{\Phi}{\sigma^2} < 0}$

in order for non-trivial solutions to exist.

So, if $\Phi > 0$ (i.e. square of ang momentum increases with radius) then we must have $\sigma^2 < 0$ and that means σ is pure imaginary: $\sigma = i\sigma_i$ (and $\sigma = -i\sigma_i$) either way we get wave motion:

$$v = \hat{v}(r)\, e^{ikz + \sigma t} = \hat{v}(r)\, e^{i(kz + \sigma_i t)}$$
$$= \hat{v}(r)\left[\cos(kz + \sigma_i t) + i\sin(kz + \sigma_i t)\right]$$

So get an axisymmetric wave that propagates axially (i.e. vertically) $\therefore$ Stable case for $\Phi > 0$

But if $\Phi < 0$ (i.e. square of ang momentum decreases with radius) then we must have $\sigma^2 > 0$. So σ is real.

Can show that $\sigma = \sigma_r$ and $\sigma = -\sigma_r$ can both occur (so σ can be pos or negative; both are possible. [Examine linearized eq^n set and consider u, v, w, p to all behave like $u = \hat{u}\, e^{i(kz) + \sigma t}$ then can see that if σ is an eigenvalue w/ eigenfunctions $\hat{u}, \hat{w}, \hat{v}, \hat{p}$ then $-\sigma$ is an eigenvalue with eigenfunctions $\hat{u}, \hat{w}, -\hat{v}, -\hat{p}$]

So sol^{ns} are of the form $e^{-\sigma_r t}$ and $e^{\sigma_r t}$. One of them blows up for sure. Uh-oh!

$\therefore$ Instability for $\Phi < 0$.

This confirms Rayleigh's criterion.

[For more details see Kundu or Chandrasekhar]