Instability of continuously-stratified shear flow

Consider a base-state consisting of a unidirectional flow in the horizontal: $\vec{U} = U(z)\,\hat{i}$, density $\bar{\rho}(z)$ and pressure $\bar{p}(z)$. There is wind shear (vertical shear of horiz wind). If shear is strong enough can get instability (Kelvin-Helmholtz instability).

See handouts for examples.

Can consider 3-D plane wave disturbances of the form:

$$\phi = \hat{\phi}(z)\, e^{i(kx + ly) - \sigma t}$$

but it turns out that two-dimensional disturbances ($l=0$) are more unstable than 3-D disturbances. Specifically, Squires Theorem states:

To each unstable 3-D disturbance there corresponds a more unstable 2-D one.

See proof in Kundu. Squire's Theorem is valid for viscous and inviscid flows though we'll only be working w/ inviscid flows.

Consider linearized Boussinesq eq^{ns} with 2-D velocity $(U+u)\,\hat{i} + w\hat{k}$, pressure $\bar{p}+p$ (hydrostatic based on $\bar{\rho}$) and density $\bar{\rho}+\rho$. So u, p, ρ are perturbation quantities, deviations from the base-state.

x-comp eq^n of motion:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

z-comp eq^n of motion:

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{g\rho}{\rho_0} - \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

where ρ_0 is a const reference density incompressibility cond^n for 2D (x,z) flow : $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

Thermo : $\frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial x} - \frac{\rho_0 N^2}{g} w = 0$

where $N^2 \equiv -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}$ (Brunt-Väisälä frequency)

Since flow is 2-D and incompressible, can introduce a streamfunction ψ through:

$$u = \frac{\partial \psi}{\partial z}, \qquad w = -\frac{\partial \psi}{\partial x}$$

Eq^{ns} of motion and thermo then become :

$$\frac{\partial^2 \psi}{\partial z \partial t} + U \frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial \psi}{\partial x} \frac{dU}{dz} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}$$

$$-\frac{\partial^2 \psi}{\partial x \partial t} - U \frac{\partial^2 \psi}{\partial x^2} = -g \frac{\rho}{\rho_0} - \frac{1}{\rho_0} \frac{\partial p}{\partial z}$$

$$\frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial x} + \rho_0 \frac{N^2}{g} \frac{\partial \psi}{\partial x} = 0$$

Coefficients are indep of x and t (but not z) $\therefore$ anticipate that spatially-wavy sol^{ns} are possible.

Seek sol^{ns} of form: $\psi = \hat{\psi}(z) e^{ik(x-ct)}$

$$p = \hat{p}(z) e^{ik(x-ct)}$$

$$\rho = \hat{\rho}(z) e^{ik(x-ct)}$$

where k is <u>real</u> and c is possibly complex ($c = c_r + i c_i$)

∴ eq^{ns} of motion and thermo become:

(1) $$(U-c)\frac{d\hat{\psi}}{dz} - \hat{\psi}\frac{dU}{dz} = -\frac{\hat{p}}{\rho_0}$$

(2) $$k^2(U-c)\hat{\psi} = -g\frac{\hat{\rho}}{\rho_0} - \frac{1}{\rho_0}\frac{d\hat{p}}{dz}$$

(3) $$(U-c)\hat{\rho} + \rho_0\frac{N^2}{g}\hat{\psi} = 0$$

eliminate $\hat{p}$ by taking $\frac{d}{dz}$ of (1):

$$\frac{dU}{dz}\frac{d\hat{\psi}}{dz} + (U-c)\frac{d^2\hat{\psi}}{dz^2} - \frac{d\hat{\psi}}{dz}\frac{dU}{dz} - \hat{\psi}\frac{d^2U}{dz^2} = -\frac{1}{\rho_0}\frac{d\hat{p}}{dz}$$

cancel

now use (2)

$$\therefore (U-c)\frac{d^2\hat{\psi}}{dz^2} - \hat{\psi}\frac{d^2U}{dz^2} - k^2(U-c)\hat{\psi} = +g\frac{\hat{\rho}}{\rho_0}$$

now use (3) to eliminate $\hat{\rho}$

$$\therefore \boxed{(U-c)\left(\frac{d^2\hat{\psi}}{dz^2} - k^2\hat{\psi}\right) - \hat{\psi}\frac{d^2U}{dz^2} + \frac{N^2}{U-c}\hat{\psi} = 0}$$

→ Taylor-Goldstein eq^n

2nd order linear variable-coefficient o.d.e.

Governs the structure of 2-D inviscid disturbances of stratified shear flow.

recall that $\psi = \hat{\psi}(z) e^{ik(x-c\tau)}$

and $c = c_r + i c_i$

$$\therefore \psi = \hat{\psi}(z) e^{ikx - ik(c_r + i c_i)\tau}$$

$$= \hat{\psi}(z) e^{ik(x - c_r\tau) - i^2 k c_i \tau} \qquad i^2 = -1$$

$$= \hat{\psi}(z) e^{ik(x - c_r\tau) + k c_i \tau}$$

$$= \hat{\psi}(z) e^{ik(x - c_r\tau)} e^{k c_i \tau}$$

So if $kc_i > 0$ we have instability.

Take complex conjugate of T.-G. eq^n:

$$(U - c^*)\left(\frac{d^2\hat{\psi}^*}{dz^2} - k^2\hat{\psi}^*\right) - \frac{d^2U}{dz^2}\hat{\psi}^* + \frac{N^2\hat{\psi}^*}{U - c^*} = 0$$

This eq^n is the same as original T.-G. eq^n but with c replaced by c^* and $\hat{\psi}$ replaced by $\hat{\psi}^*$.

This means: If $\hat{\psi}$ is an eigenfunction w/ eigenvalue c ($= c_r + i c_i$) then $\hat{\psi}^*$ is an eigenfunction w/ eigenvalue c^* ($= c_r - i c_i$)

So, for each decaying mode there is a corresponding amplifying mode. So if $c_i \neq 0$ the flow is unstable.

[Note: if the T.-G. eq^n had an "i" in it anywhere (apart from i in c) the above result would not hold].

So now we want to see what flow scenario gives us $c_i \neq 0$.

– handout on shear flow from Kundu

Change variables to:

$$\phi \equiv \frac{\hat{\psi}}{(\bar{U}-c)^{1/2}}$$

$$\therefore \hat{\psi} = (\bar{U}-c)^{1/2}\,\phi$$

$$\therefore \frac{d\hat{\psi}}{dz} = (\bar{U}-c)^{1/2}\frac{d\phi}{dz} + \frac{\frac{1}{2}\frac{d\bar{U}}{dz}\phi}{(\bar{U}-c)^{1/2}}$$

$$\therefore \frac{d^2\hat{\psi}}{dz^2} = (\bar{U}-c)^{1/2}\frac{d^2\phi}{dz^2} + \frac{\frac{1}{2}\frac{d\bar{U}}{dz}\frac{d\phi}{dz}}{(\bar{U}-c)^{1/2}}$$

$$+ \frac{\frac{1}{2}\frac{d\bar{U}}{dz}\frac{d\phi}{dz}}{(\bar{U}-c)^{1/2}} + \frac{\frac{1}{2}\frac{d^2\bar{U}}{dz^2}\phi}{(\bar{U}-c)^{1/2}} - \frac{\frac{1}{4}\left(\frac{d\bar{U}}{dz}\right)^2\phi}{(\bar{U}-c)^{3/2}}$$

combine

Plug into Taylor–Goldstein eq^n, get:

$$(\bar{U}-c)^{3/2}\frac{d^2\phi}{dz^2} + (\bar{U}-c)^{1/2}\frac{d\bar{U}}{dz}\frac{d\phi}{dz} + (\bar{U}-c)^{1/2}\frac{1}{2}\frac{d^2\bar{U}}{dz^2}\phi$$

$$- \frac{\frac{1}{4}\left(\frac{d\bar{U}}{dz}\right)^2\phi}{(\bar{U}-c)^{1/2}} - k^2(\bar{U}-c)^{3/2}\phi - (\bar{U}-c)^{1/2}\frac{d^2\bar{U}}{dz^2}\phi$$

combine

$$+ \frac{N^2}{(\bar{U}-c)^{1/2}}\phi = 0$$

$\div$ by $(U-c)^{1/2}$

$$\underbrace{(U-c)\frac{d^2\phi}{dz^2}+\frac{dU}{dz}\frac{d\phi}{dz}}_{\text{combine}}-\phi\left[\frac{1}{2}\frac{d^2U}{dz^2}+k^2(U-c)+\frac{\frac{1}{4}\left(\frac{dU}{dz}\right)^2-N^2}{U-c}\right]=0$$

$$\frac{d}{dz}\left[(U-c)\frac{d\phi}{dz}\right]-\phi\left[\frac{1}{2}\frac{d^2U}{dz^2}+k^2(U-c)+\frac{\frac{1}{4}\left(\frac{dU}{dz}\right)^2-N^2}{U-c}\right]=0$$

Mult by ϕ^* (complex conjugate of ϕ) and integrate from $z=0$ to $z=d$. Boundary conditions: in a lab put impermeable plates at $z=0$ and $z=d$. Impermeability cond^n:

$w(0)=0$
$w(d)=0$

but $w=-\frac{\partial\psi}{\partial x}$ so impermeability becomes:

$\frac{\partial\psi}{\partial x}(0)=0$ and $\frac{\partial\psi}{\partial x}(d)=0$ or, since $\psi=\hat{\psi}(z)e^{ik(x-ct)}$
$\frac{\partial\psi}{\partial x}=ik\hat{\psi}e^{ik(x-ct)}$

so $\hat{\psi}(0)=0$ and $\hat{\psi}(d)=0$ or, since $\phi\equiv\frac{\hat{\psi}}{(U-c)^{1/2}}$

so $\phi(0)=0$ and $\phi(d)=0$ now take complex conj

so $\phi^*(0)=0$ and $\phi^*(d)=0$

$$\underbrace{\int\phi^*\frac{d}{dz}\left[(U-c)\frac{d\phi}{dz}\right]dz}_{\text{integrate by parts}}-\int\phi^*\phi\left[k^2(U-c)+\frac{1}{2}\frac{d^2U}{dz^2}+\frac{\frac{1}{4}\left(\frac{dU}{dz}\right)^2-N^2}{U-c}\right]dz=0$$

integrate by parts $\longrightarrow$ $\underbrace{\left[\phi^*(U-c)\frac{d\phi}{dz}\right]_0^d}_{\text{This boundary term is 0!}}-\int\frac{d\phi^*}{dz}(U-c)\frac{d\phi}{dz}dz$

$\therefore$ get:

$$(\star) \quad -\int \left(\left|\frac{d\phi}{dz}\right|^2 (U-c) + K^2 (U-c)|\phi|^2 + \frac{1}{2}\frac{d^2U}{dz^2}|\phi|^2 + \frac{\frac{1}{4}\left(\frac{dU}{dz}\right)^2 - N^2}{U-c} |\phi|^2 \right) dz = 0$$

Note that $U-c = U - c_r - ic_i$

and $\dfrac{1}{U-c} = \dfrac{1}{U-c}\,\dfrac{U-c^*}{U-c^*} = \dfrac{1}{U-c_r-ic_i}\,\dfrac{U-c_r+ic_i}{U-c_r+ic_i}$

$$= \frac{U-c_r+ic_i}{(U-c_r)^2+c_i^2}$$

So $\operatorname{Im}(U-c) = -c_i$

and $\operatorname{Im}\left(\dfrac{1}{U-c}\right) = \dfrac{c_i}{(U-c_r)^2+c_i^2}$

Taking imaginary part of $(\star)$ yields (careful with signs!):

$$c_i \int \left(\underbrace{\left(\left|\frac{d\phi}{dz}\right|^2 + K^2|\phi|^2 \right)}_{\text{pos}} - \frac{\left(\frac{1}{4}\left(\frac{dU}{dz}\right)^2 - N^2\right)}{(U-c_r)^2+c_i^2} |\phi|^2 \right) dz = 0$$

denominator is pos $\qquad$ $|\phi|^2$ is pos

So if $\frac{1}{4}\left(\frac{dU}{dz}\right)^2 - N^2 < 0$ then $c_i \int (\text{pos})\,dz = 0$

so $c_i(\text{pos}) = 0$ which ~~means~~ can only be true if $c_i = 0$ $\therefore$ Stable case!

But if $\frac{1}{4}\left(\frac{d\bar{U}}{dz}\right)^2 - N^2 > 0$ then

$$c_i \int \left((pos) + (neg)\right) dz = 0$$

indeterminate case... we can't say if the flow is stable or unstable.

Gradient Richardson number $Ri \equiv \dfrac{N^2}{\left(\frac{dU}{dz}\right)^2}$

So if $Ri > \frac{1}{4}$ everywhere the flow is stable.

but if $Ri < 1/4$ the flow might be unstable, but who knows...

So.. $Ri < 1/4$ is a necessary but not sufficient cond^n for instability.

Note: In practice, $Ri < 1/4$ is usually unstable.

Consider special case where $N^2 = 0$ (neutral stratification). Then T.-G. eq^n reduces to:

$$\boxed{\frac{d^2\hat{\psi}}{dz^2} - k^2\hat{\psi} - \frac{\frac{d^2U}{dz^2}}{U-c}\,\hat{\psi} = 0}$$

Rayleigh's eq^n

This same eq^n (w/ y replacing z) also holds for perturbations in a flow sheared in y-dir^n $U(y)$ because gravity does not enter the problem.

Rayleigh showed that a necessary criterion for instability of an inviscid and unstratified shear flow is that the base-state velocity have an inflection point.

Proof: Multiply Rayleigh's eqⁿ by $\hat{\psi}^*$ and integrate from 0 to d (where $w=0 \rightarrow$ so $\psi=0$ so $\hat{\psi}=0$ so $\hat{\psi}^*=0$).

$$\underbrace{\int \hat{\psi}^* \frac{d^2\hat{\psi}}{dz^2}\,dz}_{} - k^2\int \hat{\psi}^*\hat{\psi}\,dz - \int\left(\frac{d^2U/dz^2}{U-c}\right)\hat{\psi}^*\hat{\psi}\,dz = 0$$

$\rightarrow$ integrate by parts, get $\underbrace{\left[\hat{\psi}^*\frac{d\hat{\psi}}{dz}\right]_0^d}_{\text{this is 0 from the b.c.}} - \int \frac{d\hat{\psi}^*}{dz}\frac{d\hat{\psi}}{dz}\,dz$

Mult by -1, get

(A) $$\int\left|\frac{d\hat{\psi}}{dz}\right|^2 dz + k^2\int|\hat{\psi}|^2 dz + \int\left(\frac{\frac{d^2U}{dz^2}}{U-c}\right)|\hat{\psi}|^2 dz = 0$$

recall that $\frac{1}{U-c} = \frac{U-c_r+ic_i}{(U-c_r)^2+(c_i)^2}$ so the imaginary part of above eqⁿ is:

$$c_i\int\frac{\frac{d^2U}{dz^2}}{(U-c_r)^2+(c_i)^2}|\hat{\psi}|^2\,dz = 0$$

Suppose flow is unstable (so $c_i \neq 0$). Then must have

$$\int\frac{\frac{d^2U}{dz^2}}{(U-c_r)^2+(c_i)^2}|\hat{\psi}|^2\,dz = 0$$

pos

So $\frac{d^2U}{dz^2}$ must be pos and neg within the flow domain $\therefore$ $\frac{d^2U}{dz^2}$ must be 0 (inflection point) somewhere in interior. So an inflection point in $U(z)$ within the flow domain is a necessary cond^n for instability (but it's not a sufficient cond^n).

To get a stronger necessary cond^n for instability, Fjortoft also considered the real part of (A). Regardless of whether c_i is 0 or not, the real part of (A) is:

$$\underbrace{\int \left|\frac{d\hat{\psi}}{dz}\right|^2 dz + k^2 \int |\hat{\psi}|^2 dz}_{\text{this is pos}} + \underbrace{\int \frac{\frac{d^2U}{dz^2}(U - c_r)}{(U-c_r)^2 + (c_i)^2} |\hat{\psi}|^2 dz}_{\therefore \text{ this must be neg}} = 0$$

$$\therefore \quad (B) \quad \int \frac{\frac{d^2U}{dz^2}(U - c_r)}{(U-c_r)^2 + (c_i)^2} |\hat{\psi}|^2 dz < 0$$

Now again consider unstable case ($c_i \neq 0$). Multiply Rayleigh's inflection point criterion by $c_r - U_I$ where U_I is the base-state velocity at the inflection point.

$$\therefore \quad \int \frac{\frac{d^2U}{dz^2}(c_r - U_I)}{(U-c_r)^2 + (c_i)^2} |\hat{\psi}|^2 dz = 0$$

Adding this result to the inequality (B), we get:

$$\int \frac{\frac{d^2U}{dz^2}(U - U_I)}{(U-c_r)^2 + (c_i)^2} |\hat{\psi}|^2 dz < 0$$

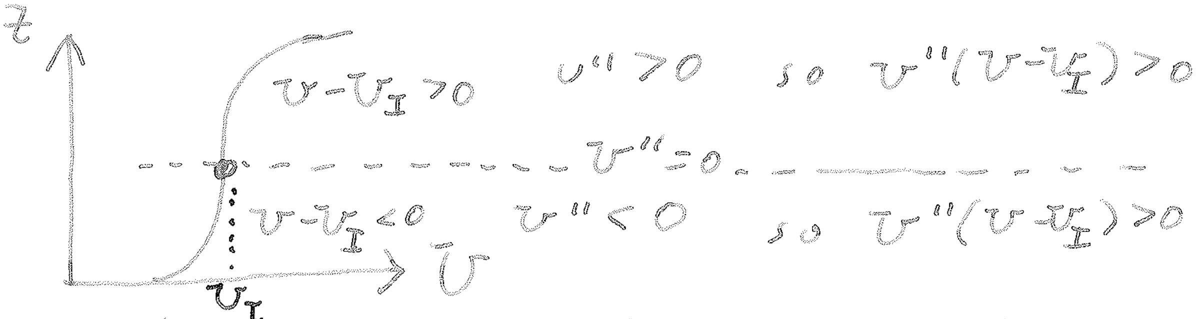
So in the unstable case must have:

$$\frac{d^2 U}{dz^2}\,(U - U_I) < 0 \text{ somewhere in the flow.}$$

It's a necessary cond^n for instability. This is Fjortoft's Theorem.

Another way to state Fjortoft's Th^m: A necessary cond^n for instability of inviscid shear flow is that the magnitude of the vorticity of the base-state U field has a maximum within the flow domain (i.e. not good enough to have it just on boundaries).

Rayleigh's inflection criterion implies that there be an extremum in the vorticity magnitude (max or min). Fjortoft refines (strengthens) the criterion by saying the extremum must be a maximum.



In above diagram there is an inflection point but it's associated with a minimum in the magnitude of vorticity. Fjortoft's Th^m says: no instability in this case.

→ Go through handout from Kundu on parallel shear flows.

(a) stable (b) stable
(c) stable (d) Rayleigh: might be unstable; Fjortoft: stable (wins)
(e) Rayleigh + Fjortoft both: might be unstable (f) same as (e).