The Boussinesq Approximation in Stratified Fluids

METR 5123 Advanced Atmospheric Dynamics II

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The Boussinesq version of the Navier-Stokes equations appears in the literature in different forms. Some of the differences are cosmetic (notational differences), but others arise from different partitions of the pressure and density fields. Two such partitions are in common usage. We'll look at both of them.

The Boussinesq approximation is a set of approximations. The approximation(s) is typically applied to low Mach-number convective flows where density variations are "sufficiently small". The first two of these approximations are that material properties such as μ are considered constant, and that the mass conservation equation can be safely replaced by the incompressibility condition $\nabla \cdot \vec{u} = 0$. These two approximations reduce the Navier-Stokes equations to the form

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla p - g\hat{k} + \frac{\mu}{\rho}\nabla^2 \vec{u}. \qquad (1)$$

Here $\hat{k}$ is the unit vector in the vertical direction, p is pressure, ρ is density, and other symbols have their conventional meaning. Multiplying (1) by ρ yields

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}\right] = -\nabla p - \rho g\hat{k} + \mu\nabla^2 \vec{u}. \qquad (2)$$

The third of these Boussinesq approximations can be summarized as: "Density differences are sufficiently small that they can be neglected, except where they appear in terms multiplied by g." So, when we invoke this part of the Boussinesq approximation, the density ρ in the inertia term (left hand side of (2)), is replaced by a constant reference value ρ_0, and (2) becomes

$$\rho_0\left[\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}\right]=-\nabla p-\rho g\hat{k}+\mu\nabla^2\vec{u}. \tag{3}$$

Dividing (3) by ρ_0, we obtain

$$\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}=-\frac{1}{\rho_0}\nabla p-\frac{\rho}{\rho_0}g\hat{k}+\nu\nabla^2\vec{u}, \tag{4}$$

where $\nu\equiv\mu/\rho_0$ is constant since μ and are ρ_0 constants. And that's it. We may regard (4) as the Navier-Stokes equations under the Boussinesq approximation. However, a check of the literature shows that the Navier-Stokes equations under the Boussinesq approximation are often expressed in alternative forms. Specifically, the p and ρ fields are partitioned into two terms each, generically referred to as "base-state" and "perturbation" components, and the first two terms on the right hand side of (4) are rewritten accordingly. As we will see, the partitioning/rewriting steps to be discussed next do not introduce any further approximation to the Navier-Stokes equations and so the rewritten forms are entirely equivalent to (4).

Two different ways of partitioning each of p and ρ are in common usage. Of these two partitions – or any other partition of p or ρ – there is no right or wrong way unless a further approximation, beyond the already-imposed Boussinesq approximations, is applied. However, the interpretation of the individual terms will be different between partitions. The situation is analogous to partitioning the number 9 as 9 = 8 + 1 or as 9 = 5 + 4. Both partitions are valid. However, if, for convenience, it were desirable to neglect the smaller of the two terms in one of the partitions, it would be better to consider the first partition.

We may decompose ρ into a height-dependent base-state profile $\overline{\rho}(z)$ (also referred to as a mean, reference, environmental or sounding profile – the various names suggesting that there may be some ambiguity in the way such a profile would be arrived at in practice) and the perturbation density $\rho'(x,y,z,t)$ defined to be the deviation of ρ from the base-state profile $\overline{\rho}(z)$.

We then define a base-state pressure distribution $\bar{p}(z)$ to be the pressure in a virtual (hypothetical) atmosphere that is hydrostatic and has a density field equal to the base-state density $\bar{\rho}(z)$. The perturbation pressure $p'(x,y,z,t)$ is then defined to be the deviation of the actual pressure field from this base-state pressure field. This partition can be written as

$$\rho(x,y,z,t) = \bar{\rho}(z) + \rho'(x,y,z,t), \qquad p(x,y,z,t) = \bar{p}(z) + p'(x,y,z,t), \tag{5}$$

where $\bar{p}$ satisfies

$$\frac{d\bar{p}}{dz} = -\bar{\rho}(z)\, g. \tag{6}$$

Applying (5) in (4) and making use of (6), we obtain

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0} g\hat{k} + \nu\nabla^2\vec{u}\ , \tag{7}$$

which is one of the conventional forms of the Navier-Stokes equations in the Boussinesq approximation. The second term on the right hand side of (7) is the conventional buoyancy variable for Boussinesq flows:

$$b \equiv -\frac{\rho'}{\rho_0} g = -g\frac{\rho - \bar{\rho}(z)}{\rho_0}. \tag{8}$$

This differs slightly from non-Boussinesq definitions of buoyancy [e.g., $-g\dfrac{\rho - \bar{\rho}(z)}{\bar{\rho}(z)}$], which has a height-varying base-state density in the denominator as well as in the numerator. Consider a stratified fluid at rest. Then something generates a disturbance. But assume that the notion of an "environment" still exists, so we can always move far enough away from the disturbance to be where ρ is equal to $\bar{\rho}$. So, far from the disturbance, the Boussinesq buoyancy defined by (8) goes to zero. Also, far from the disturbance the velocity field goes to zero (environment was considered to be at rest). So, from (7), we see that far from the disturbance, the perturbation pressure gradient $\nabla p'$ must go to zero.

Now consider a partition in which ρ is written as the sum of the constant reference value ρ_0 and a perturbation component $\rho''(x,y,z,t)$ defined to be the deviation of ρ from ρ_0. We then define a base-state pressure distribution $\bar{\bar{p}}(z)$ to be the pressure in a virtual atmosphere that is hydrostatic and has a density field equal to ρ_0. The perturbation pressure $p''(x,y,z,t)$ is then defined to be the deviation of p from $\bar{\bar{p}}(z)$. We can write this second type of partition as:

$$\rho(x,y,z,t) = \rho_0 + \rho''(x,y,z,t)\,, \qquad p(x,y,z,t) = \bar{\bar{p}}(z) + p''(x,y,z,t)\,, \tag{9}$$

where $\bar{\bar{p}}$ satisfies

$$\frac{d\bar{\bar{p}}}{dz} = -\rho_0\, g\,. \tag{10}$$

In terms of these new base-state and perturbation variables, (4) becomes

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho_0}\nabla p'' - \frac{\rho''}{\rho_0} g\hat{k} + \nu\nabla^2\vec{u}\,. \tag{11}$$

If in (11) the symbols p' and ρ' are used in place of p'' and ρ'', then (11) would be identical to (7). However, the perturbation pressure gradient and buoyancy terms in (11) and (7) are generally not the same – they arise from two different partitions. To see the relation between the two perturbation pressure gradient terms, use (5), (6), (9), (10) to write

$$-\frac{1}{\rho_0}\nabla p'' = -\frac{1}{\rho_0}\nabla(p - \bar{\bar{p}}) = -\frac{1}{\rho_0}\nabla(p' + \bar{p} - \bar{\bar{p}}) = -\frac{1}{\rho_0}\nabla p' + g\,\frac{\bar{\rho} - \rho_0}{\rho_0}\,. \tag{12}$$

The buoyancy variable $-\frac{\rho''}{\rho_0} g\hat{k}$ appearing in (11) differs from the buoyancy variable defined by (8). Now consider the same scenario considered above, of a disturbance developing within a stratified fluid otherwise at rest. Again, far from the disturbance, ρ becomes equal to $\bar{\rho}$.

However, this means that far from the disturbance the buoyancy variable $-\frac{\rho''}{\rho_0} g\hat{k}$ becomes equal to $-g\frac{\overline{\rho}(z)-\rho_0}{\rho_0}$, which is generally not zero! This is not an error as long as the perturbation pressure gradient force $-\frac{1}{\rho_0}\nabla p''$ (which, as we've just seen, contains a residual hydrostatic component) is retained. So, in the environment far from the disturbance, neither $-\frac{1}{\rho_0}\nabla p''$ nor $-\frac{\rho''}{\rho_0} g\hat{k}$ are zero, but they sum to zero (easy to show that they sum to $-\frac{1}{\rho_0}\nabla p'$, which we've shown is zero). Thus, although the interpretation of the individual terms in this second partition may not be as appealing as in the first partition, it does yield the correct result.

The two different partitions and two different Boussinesq forms of the Navier-Stokes equations appearing in the literature has led to an unfortunate situation in which the same symbols and terminology, e.g., "perturbation pressure", "perturbation density" and the related quantity "buoyancy" have been applied to different terms. To reduce the potential for confusion and misconceptions, authors (and readers) should be clear on the precise definitions being considered for these terms.