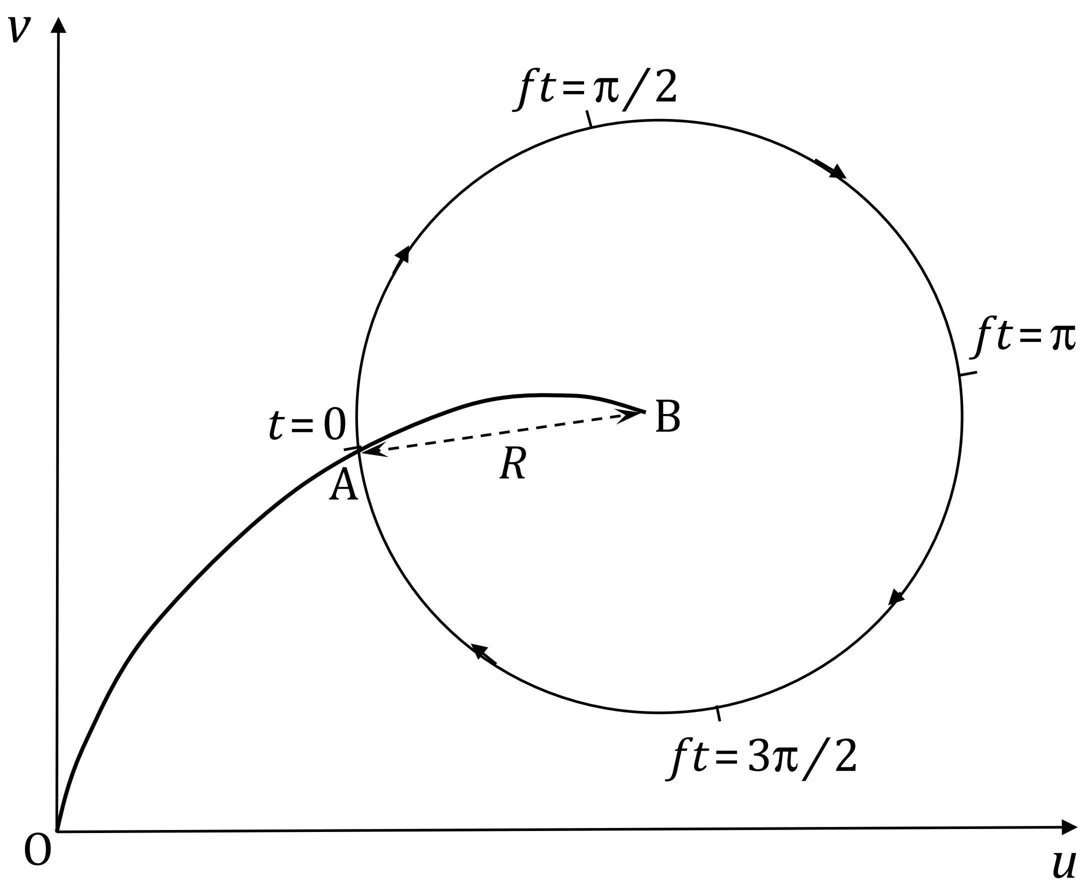
A Blackadar-like theory for the nocturnal low-level jet



Schematic of an air parcel's Cartesian (u, v) velocity components in a Northern hemisphere inertial oscillation (adapted from Blackadar, 1957). Curve OAB is the initial (t = 0) hodograph, roughly at the time of sunset. Point O is at ground level. Point B is at the top of boundary layer, where the flow is considered to be geostrophic. Point A is an arbitrary location on the initial hodograph. An air parcel released from the frictional constraint at t = 0 undergoes an inertial oscillation, manifested on the hodograph plane as a circle with radius R equal to the magnitude of the parcel's initial ageostrophic wind speed.

Blackadar-like models

- Buajitti & Blackadar (1957), Singh *et al.* (1993), and Tan & Farahani (1998) considered viscous models but with a slowly-varying K (sine with a period of a day).

- Thorpe & Guymer (1977) considered 3-layer slab-models with empirical stress relations imposed in the lower layer for $t > 0$.

- Shapiro & Fedorovich (2010) theory: viscous model with abrupt change (reduction) in eddy-viscosity at $t = 0$.

Equations of motion

In a Cartesian (x, y) coordinate system with *x-axis aligned with the geostrophic wind vector*, the Navier-Stokes equations become:

$$\frac{\partial u}{\partial t} = f\,v + K\frac{\partial^2 u}{\partial z^2}, \qquad (1)$$

$$\frac{\partial v}{\partial t} = -f\left(u - u_G\right) + K\frac{\partial^2 v}{\partial z^2}. \qquad (2)$$

f: Coriolis parameter

u, v: x- and y- velocity components, respectively

K: Viscosity coefficient (eddy-viscosity), assumed constant

u_g: Geostrophic wind, assumed constant

Boundary conditions

Impose no-slip conditions at the ground,

$$\begin{cases} u(0,t) = 0, \\ v(0,t) = 0, \end{cases} \tag{3}$$

and pure geostrophic flow far above the ground:

$$\begin{cases} \lim_{z\to\infty} u(z,t) = u_G, \\ \lim_{z\to\infty} v(z,t) = 0. \end{cases} \tag{4}$$

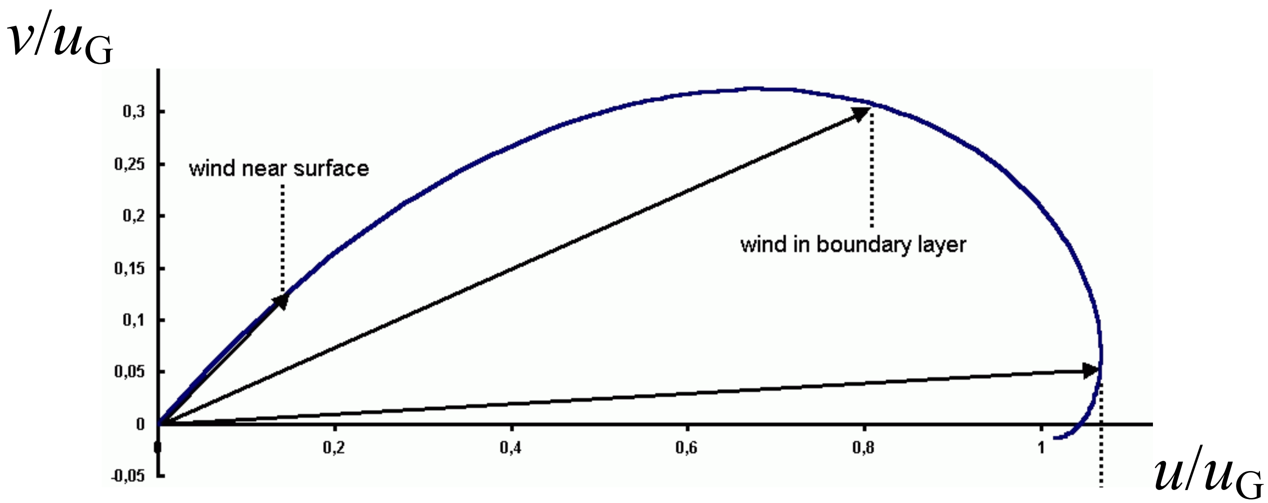
Initial conditions

Obtain the initial velocity field from the steady state version of (1) and (2) with eddy viscosity K_0 (> K),

$$0 = f\,v(z,0) + K_0 \frac{d^2u(z,0)}{dz^2}, \tag{5}$$

$$0 = -f\left[u(z,0) - u_G\right] + K_0 \frac{d^2v(z,0)}{dz^2}. \tag{6}$$

The solution of (5) and (6) yields the classical Ekman spiral:



Non-dimensionalization

$Z \equiv z\sqrt{\frac{f}{K_0}},$ $Z = 1$ is roughly the initial Ekman depth

$T \equiv f\,t,$ $T = 2\pi$ is the inertial period

$U \equiv \frac{u}{u_G},,$ $U \rightarrow 1$ as $Z \rightarrow \infty$

$V \equiv \frac{v}{u_G},$ $V \rightarrow 0$ as $Z \rightarrow \infty$

$\varepsilon \equiv \frac{K}{K_0},$ ε is a turbulence reduction parameter

ε << 1 but how small is it? 10^{-1}? 10^{-2}? 10^{-9}? Banta (2008): "turbulent fluxes …were found to be essentially shut down".

The non-dimensional problem

$$\frac{\partial U}{\partial T}=V+\varepsilon\frac{\partial^2 U}{\partial Z^2}, \tag{7}$$

$$\frac{\partial V}{\partial T}=1-U+\varepsilon\frac{\partial^2 V}{\partial Z^2}, \tag{8}$$

$$U(0,T)=0, \quad V(0,T)=0, \tag{9}$$

$$\lim_{Z\to\infty} U(Z,T)=1, \quad \lim_{Z\to\infty} V(Z,T)=0, \tag{10}$$

$$0=V(Z,0)+\frac{d^2U(Z,0)}{dZ^2}, \tag{11}$$

$$0=1-U(Z,0)+\frac{d^2V(Z,0)}{dZ^2}. \tag{12}$$

This problem has only 1 degree of freedom: ε.

A trick to simplify the problem

The trick in question is commonly used to solve certain types of coupled linear ODEs and PDEs. Unfortunately the trick is often used without explanation (e.g. Holton's derivation of the Ekman solution). Here you'll see where the trick comes from.

First write (7) and (8) as

$$\frac{\partial}{\partial T}\left(U-1\right)=V+\varepsilon\frac{\partial^2}{\partial Z^2}\left(U-1\right), \tag{7'}$$

$$\frac{\partial V}{\partial T}=-\left(U-1\right)+\varepsilon\frac{\partial^2 V}{\partial Z^2}. \tag{8'}$$

Next, multiply (8') by a constant c, and add the result to (7').

$$\frac{\partial}{\partial T}\left(U-1+cV\right)=V-c\left(U-1\right)+\varepsilon\frac{\partial^2}{\partial Z^2}\left(U-1+cV\right).$$

The first and last terms combine U and V into a new variable:

$$\Phi\equiv U-1+cV.$$

The trick continued

So now the combined PDE becomes

$$\frac{\partial\Phi}{\partial T} = V - c\left(U-1\right) + \varepsilon\frac{\partial^2\Phi}{\partial Z^2}.$$

Lets see if we can rewrite the middle term $V - c(U-1)$ so that it's proportional to Φ, that is, we want to write $V - c(U-1) = \beta\Phi$, where β is another constant. Since $\Phi \equiv U - 1 + cV$, we must have:

$$V - c(U-1) = \beta\left[(U-1) + cV\right]$$

Rearrange it as

$$(1-\beta c)V - (c+\beta)(U-1) = 0$$

For this equation to be satisfied (for all Z), the coefficients of V and $U-1$ must vanish. So we must have

$$c+\beta = 0, \quad \text{and} \quad 1-\beta c = 0.$$

The trick completed

From $c+\beta=0$ we get $c=-\beta$. Plugging this into $1-\beta c=0$, we get:

$$1+\beta^2=0$$

$$\therefore\ \beta^2=-1$$

$\therefore\ \beta=\pm i$. Choice of sign doesn't matter. We only need one β.

So $\beta=-i$, $c=i$, $\Phi\equiv U-1+iV$ and the combined PDE becomes

$$\boxed{\frac{\partial\Phi}{\partial T}=-i\Phi+\varepsilon\frac{\partial^2\Phi}{\partial Z^2}}$$

We have thus tricked two coupled PDEs in two dependent variables U, V to collapse into one PDE for a new variable Φ.

But in order for this trick to be useful, it must also be possible to collapse the boundary and initial conditions into conditions on Φ. Fortunately, this turns out to be the case.

The simplified initial boundary value problem

In terms of the single variable

$$\Phi \equiv U - 1 + iV, \tag{13}$$

equations (7)-(12) reduce to

$$\frac{\partial \Phi}{\partial T} = -i\Phi + \varepsilon \frac{\partial^2 \Phi}{\partial Z^2}, \tag{14}$$

$$\Phi(0, T) = -1 \tag{15}$$

$$\lim_{Z \to \infty} \Phi(Z, T) = 0, \tag{16}$$

$$\Phi(Z, 0) = -\exp\left[-(1+i)\, Z/\sqrt{2}\right]. \tag{17}$$

Equation (17) is the solution of (11) and (12) (the classical Ekman solution) recast in terms of Φ.

Laplace-transformed equation set

Taking the Laplace transform L of (14)-(16), and using (17) yields the ordinary differential equation (ODE):

$$\varepsilon \frac{d^2 F}{dZ^2} - (s+i) F = \exp\left[-(1+i) Z/\sqrt{2}\right], \tag{19}$$

where $F \equiv \int_0^\infty \exp(-sT)\Phi(Z,T)\,dT$. This ODE is subject to the boundary conditions:

$$F(0) = -\frac{1}{s}, \tag{20}$$

$$\lim_{Z\to\infty} F(Z) = 0. \tag{21}$$

Solution of the ODE for *F*

The homogeneous solution of (19) is

$$F_h = A\exp\left(\sqrt{\frac{s+i}{\varepsilon}}\,Z\right) + B\exp\left(-\sqrt{\frac{s+i}{\varepsilon}}\,Z\right), \tag{22}$$

and a particular solution is found by inspection to be a constant times the exponential term in (19). Apply boundary conditions (20) and (21) to get *A* and *B*. We thus obtain the solution for *F* as:

$$F = -\frac{1}{s-i(\varepsilon-1)}\exp\left[-\frac{(1+i)}{\sqrt{2}}Z\right] + \frac{i(\varepsilon-1)}{s\left[s-i(\varepsilon-1)\right]}\exp\left(-\sqrt{\frac{s+i}{\varepsilon}}\,Z\right). \tag{23}$$

Inverse transformation

Get $\Phi = L^{-1}(\mathrm{F})$ by evaluating the inverse Laplace transform L^{-1} of (23). Using standard theorems (similarity, shifting, convolution) and tabulated results, we obtain the solution in closed form as:

$$\Phi(Z,T) = -\exp\left[-\frac{(1+i)}{\sqrt{2}}Z + i(\varepsilon-1)T\right] - \int_0^T \frac{Z}{2\sqrt{\pi\varepsilon}\,\tau^{3/2}} \exp\left(-i\tau - \frac{Z^2}{4\varepsilon\tau}\right) d\tau$$

$$+\exp\left[-i(1-\varepsilon)T\right]\int_0^T \frac{Z}{2\sqrt{\pi\varepsilon}\,\tau^{3/2}} \exp\left(-i\varepsilon\tau - \frac{Z^2}{4\varepsilon\tau}\right) d\tau. \qquad (24)$$

An equivalent series solution

Since we are mostly interested in times on the order of a ½-period of an inertial oscillation ($T \sim \pi$), a convenient means of evaluating (24) can follow from appropriate Taylor expansions about $T = 0$.

Expanding exp(-$i\tau$) and exp(-$i\varepsilon\tau$) in the two integrands yields

$$\Phi(Z,T) = -\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^T \frac{\tau^{n-3/2}}{2\sqrt{\pi\varepsilon}} Z \exp\left(-\frac{Z^2}{4\varepsilon\tau}\right) d\tau \left\{1 - \varepsilon^n \exp\left[-i\left(1-\varepsilon\right)T\right]\right\}$$

$$-\exp\left[-\frac{(1+i)}{\sqrt{2}} Z + i\left(\varepsilon - 1\right)T\right]. \qquad (25)$$

Evaluating the integral

Changing variables in the integrand of (25) to $\xi \equiv Z/(2\sqrt{\varepsilon\tau})$ yields

$$\Phi(Z,T) = \sum_{n=0}^{\infty} \frac{I(Z,T;n)}{n!}\left\{\varepsilon^n \exp\left[-i\left(1-\varepsilon\right)T - i\frac{n\pi}{2}\right] - \exp\left(-i\frac{n\pi}{2}\right)\right\}$$

$$-\exp\left[-\frac{(1+i)}{\sqrt{2}}Z + i\left(\varepsilon-1\right)T\right], \qquad (26)$$

where

$$I(Z,T;n) \equiv \left(\frac{Z}{2\sqrt{\varepsilon}}\right)^{2n} \frac{2}{\sqrt{\pi}} \int_{\frac{Z}{2\sqrt{\varepsilon T}}}^{\infty} \xi^{-2n} \exp\left(-\xi^2\right) d\xi. \qquad (27)$$

Recursive formula for $I(Z, T; n)$

Integrating (27) by parts yields a recursive solution for $I(Z,T;n)$

involving the complementary error function:

$$I(Z,T;n)=\begin{cases} \operatorname{erfc}\left(\dfrac{Z}{2\sqrt{\varepsilon T}}\right), & n=0, \\[2ex] \dfrac{Z}{\sqrt{\pi\,\varepsilon}}\dfrac{T^{n-1/2}}{2n-1}\exp\left(-\dfrac{Z^2}{4\varepsilon T}\right)-\dfrac{Z^2}{2\varepsilon}\dfrac{I(Z,T;n-1)}{2n-1}, & n=1,2,3\ldots \end{cases} \tag{28}$$

Finally we get *U* and *V*

Since $U = 1+\mathrm{Re}(\Phi)$ and $V = \mathrm{Im}(\Phi)$, (26) yields

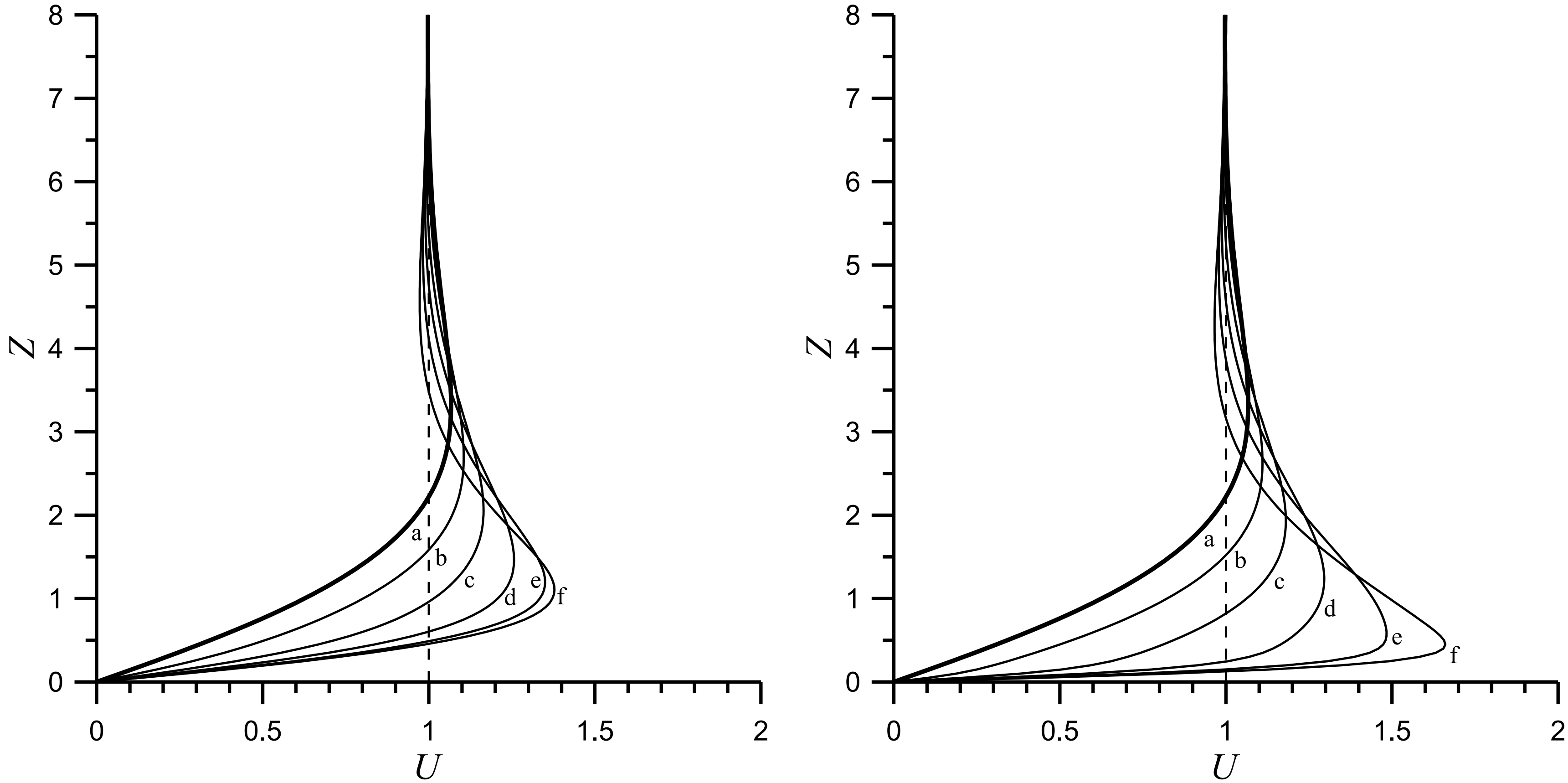
$$U = \sum_{n=0}^{\infty} \frac{I(Z,T;n)}{n!}\left\{\varepsilon^n \cos\left[\left(1-\varepsilon\right)T + \frac{n\pi}{2}\right] - \cos\left(\frac{n\pi}{2}\right)\right\}$$

$$+1-\exp\left(-\frac{Z}{\sqrt{2}}\right)\cos\left[\frac{Z}{\sqrt{2}}+\left(1-\varepsilon\right)T\right], \qquad (29)$$

$$V = \sum_{n=0}^{\infty} \frac{I(Z,T;n)}{n!}\left\{\sin\left(\frac{n\pi}{2}\right) - \varepsilon^n \sin\left[\left(1-\varepsilon\right)T + \frac{n\pi}{2}\right]\right\}$$

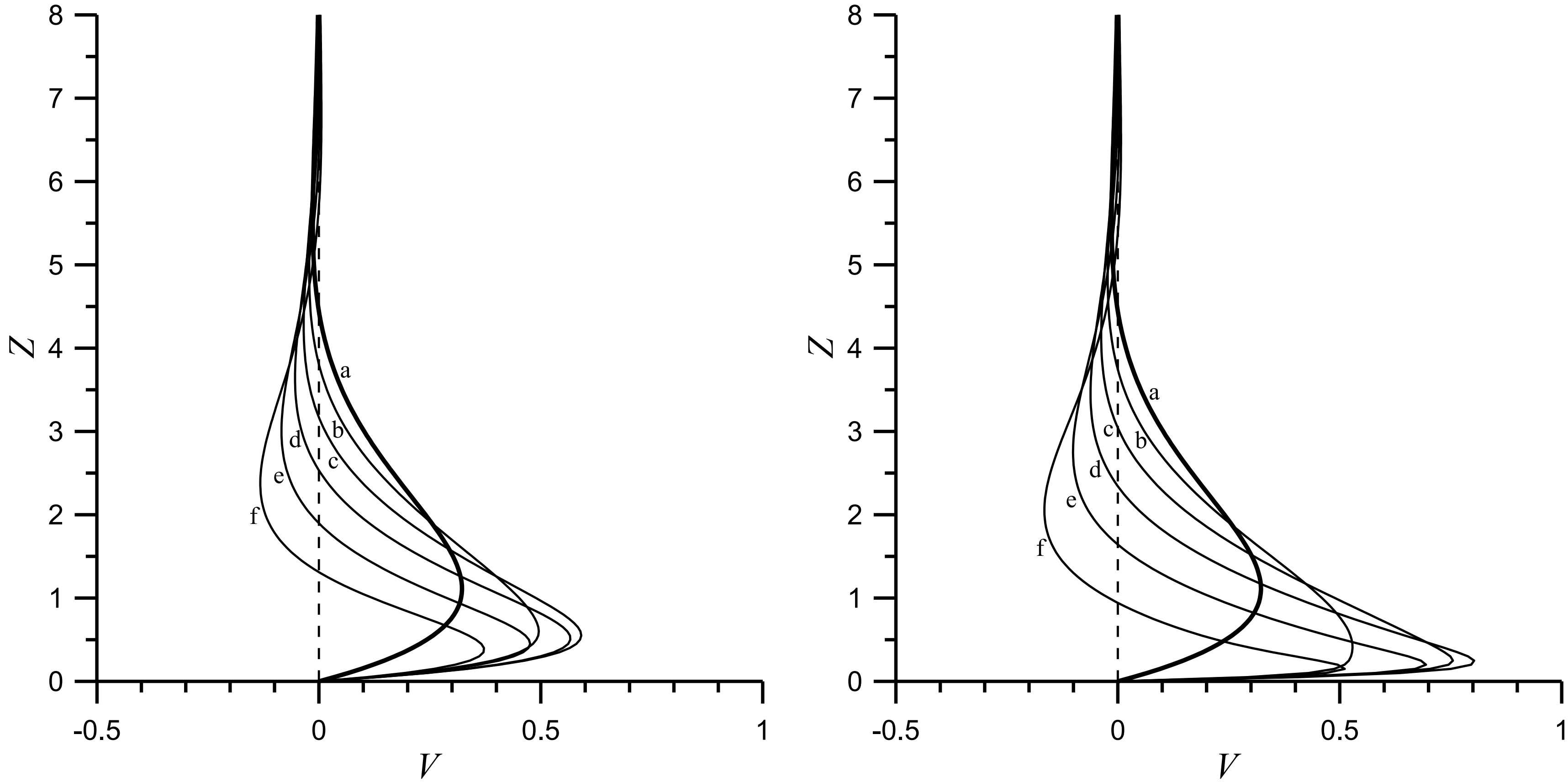
$$+\exp\left(-\frac{Z}{\sqrt{2}}\right)\sin\left[\frac{Z}{\sqrt{2}}+\left(1-\varepsilon\right)T\right], \qquad (30)$$

Evolution of the *U* profile



Evolution of vertical profiles of U for $\varepsilon = 0.1$ (left panels) and $\varepsilon = 0.01$ (right panels). Curves a, b, c, d, e, f correspond to times $T = 0, 0.5, 1, 1.5, 2, 2.5$, respectively. Vertical dashed lines denotes geostrophic wind.

Evolution of the *V* profile



Evolution of vertical profiles of V for $\varepsilon = 0.1$ (left panels) and $\varepsilon = 0.01$ (right panels). Curves a, b, c, d, e, f correspond to times $T = 0, 0.5, 1, 1.5, 2, 2.5$, respectively.