REVIEW OF SMALL SCALE INTERNAL GRAVITY WAVES IN A CONTINUOUSLY STRATIFIED FLUID

- Suppose mean density $\bar{\rho}(z)$ decreases continuously w/ height.
- Neglect friction, and Coriolis force.
- Work with perturbation quantities:

$$\rho = \bar{\rho}(z) + \rho'(x,y,z,t) \;\rightarrow\; \rho' = \rho - \bar{\rho}$$

$$p = \bar{p}(z) + p'(x,y,z,t) \;\rightarrow\; p' = p - \bar{p}$$

where $\bar{\rho}(z)$ is the environmental density, and $\bar{p}(z)$ is the environmental pressure, considered to be hydrostatic and based on $\bar{\rho}$, that is: $d\bar{p}/dz = -\bar{\rho}g$

Work w/ linearized inviscid Boussinesq eq^{ns} of motion,

$$(1) \quad \frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$(2) \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$(3) \quad \frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0}$$

incomp cond^{n}:

$$(4) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where ρ_0 is a constant reference density.

<u>Thermodynamic energy eqⁿ</u> for a non-diffusive liquid:

$$\frac{D\rho}{Dt} = 0 .$$

Rewrite it in terms of base-state and perturbation densities:

$$\therefore \quad \frac{\partial(\bar{\rho}+\rho')}{\partial t} + u\frac{\partial(\bar{\rho}+\rho')}{\partial x} + v\frac{\partial(\bar{\rho}+\rho')}{\partial y} + w\frac{\partial(\bar{\rho}+\rho')}{\partial z} = 0$$

$$\therefore \quad \frac{\partial\rho'}{\partial t} + u\frac{\partial\rho'}{\partial x} + v\frac{\partial\rho'}{\partial y} + w\frac{\partial\bar{\rho}}{\partial z} + w\frac{\partial\rho'}{\partial z} = 0$$

linearize it, get:

$$(5) \quad \frac{\partial\rho'}{\partial t} + w\frac{\partial\bar{\rho}}{\partial z} = 0$$

Eqns (1) - (5) are 5 eq^{ns} in 5 unknowns. Let's get one eq^{n} for just w. Start by eliminating u, v. Take ∂/∂x (1) + ∂/∂y (2):

$$\frac{\partial}{\partial t}\boxed{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial x^2} - \frac{1}{\rho_0}\frac{\partial^2 p'}{\partial y^2}$$

from (4) it's – ∂w/∂z

$$(6) \quad \nabla_H^2 p' = \rho_0 \frac{\partial^2 w}{\partial t\,\partial z}$$

To eliminate ρ', take ∂/∂t (3):

$$\frac{\partial^2 w}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial t\,\partial z} - \frac{g}{\rho_0}\boxed{\frac{\partial\rho'}{\partial t}}$$

→ – w dρ̄/dz from (5)

$$\frac{\partial^2 w}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial t\,\partial z} + \frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}\,w$$

Define Brunt-Väisällä frequency $N^2(z) \equiv -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}$

$$(7)\quad \frac{1}{\rho_0}\frac{\partial^2 p'}{\partial t\,\partial z} = -\frac{\partial^2 w}{\partial t^2} - N^2 w$$

Eqns (6) and (7) are two eqns in two unknowns. To eliminate p' from (6) and (7), take ∇_H^2 (7):

$$\frac{1}{\rho_0}\frac{\partial^2}{\partial t\partial z}\boxed{\nabla_H^2 p'} = -\frac{\partial^2}{\partial t^2}\nabla_H^2 w - N^2\nabla_H^2 w$$

$\downarrow$

$\rho_0 \frac{\partial^2 w}{\partial t\,\partial z}$ from (6)

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2 w}{\partial z^2} = -\frac{\partial^2}{\partial t^2}\nabla_H^2 w - N^2\nabla_H^2 w$$

use $\nabla^2 = \nabla_H^2 + \frac{\partial^2}{\partial z^2}$

$$\boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2(z)\nabla_H^2 w = 0}$$

<u>Internal wave eq$^{\underline{n}}$.</u> 4th order linear homogeneous PDE.

Lets examine simplest case: <u>N(z) = const.</u> Another way to think of it: we're considering waves with wavelengths that are <u>much smaller than the scale over which N(z) changes appreciably</u>. So, as far as these waves are concerned, N is pretty much constant for them.

Now the pde has const coefficients. Try a plane wave solution:

$$w = w_0 e^{i(kx + ly + mz - \omega t)} = w_0 e^{i(\vec{K}\cdot\vec{x} - \omega t)}$$

$$\vec{K} \equiv k\hat{i} + l\hat{j} + m\hat{k} \quad \text{is wavenumber vector.}$$

$$\vec{k}_H \equiv k\hat{i} + l\hat{j} \quad \text{is horizontal wavenumber vector.}$$

$$\vec{K} = \vec{k}_H + m\hat{k}$$

$$\left|\vec{K}\right| = \sqrt{k^2 + l^2 + m^2}\,.$$

$$k_H \equiv \left|\vec{k}_H\right| = \sqrt{k^2 + l^2}\,.$$

Plug expression for w into internal wave eq^n, get:

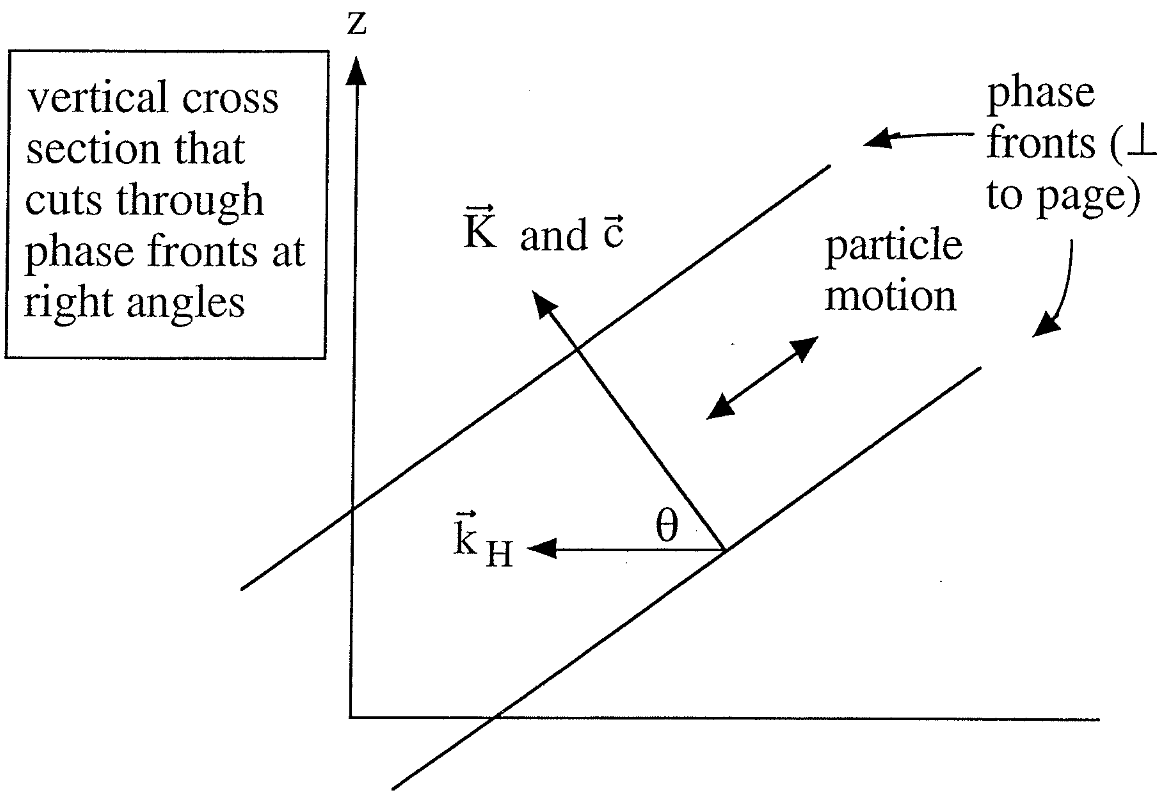
$$\therefore \quad (i\omega)^2(-k^2 - l^2 - m^2) - N^2(k^2 + l^2) = 0$$

$$\therefore \quad \left|\vec{K}\right|^2\omega^2 - N^2 k_H^2 = 0$$

$$\omega = \sqrt{N^2 \frac{k_H^2}{\left|\vec{K}\right|^2}}$$

If $d\bar{\rho}/dz < 0$ (statically stable case) then $N^2 > 0$ and $\omega = Nk_H/\left|\vec{K}\right|$ is a real number. So $e^{i(\vec{K}\cdot\vec{x} - \omega t)}$ is a propagating wave.

On the other hand, if $d\bar{\rho}/dz > 0$ (statically unstable case), then ω is imaginary, and $e^{i(\vec{K}\cdot\vec{x} - \omega t)}$ blows up exponentially with t.



θ is angle between $\vec{K}$ and horizontal.

$\therefore \; k_H = \left|\vec{K}\right| \cos\theta$

$\therefore \; \boxed{\omega = N\cos\theta}$ dispersion relation.

Frequency depends on dir^n of wavenumber (in vertical plane), but <u>not</u> on magnitude of wavenumber -- waves are <u>anisotropic</u> (care about direction). [In contrast, sfc gravity waves or internal waves on an inversion propagate horizontally with freq that depends on mag of wavenumber.]

Can show u and v are of the form:

$$u = u_0 e^{i(kx + ly + mz - \omega t)}, \qquad v = v_0 e^{i(kx + ly + mz - \omega t)}$$

Plug into incomp cond^n:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$\therefore \quad iku + ilv + imw = 0$

$\therefore \quad \vec{K} \cdot \vec{u} = 0$ So $\vec{u}$ is $\perp$ to wavenumber vector ($\vec{u}$ is parallel to phase fronts). - "shear wave", "transverse wave"

Max possible frequency is N, occurs when $\theta = 0$ -- vertical phase fronts, get up/down parcel motion.

Calculate group velocity: $\vec{c}_g = \frac{\partial \omega}{\partial k}\hat{i} + \frac{\partial \omega}{\partial l}\hat{j} + \frac{\partial \omega}{\partial m}\hat{k}$

$$\frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}\left(N\frac{k_H}{|\vec{K}|}\right) = \text{painful mess} = \frac{N k m^2}{k_H |\vec{K}|^3}$$

Similarly: $\frac{\partial \omega}{\partial l} = \frac{N l m^2}{k_H |\vec{K}|^3}$ and $\frac{\partial \omega}{\partial m} = = -\frac{Nm k_H}{|\vec{K}|^3}$

Put 'em all together, get:

$$\boxed{\vec{c}_g = \frac{Nm}{|\vec{K}|^3}\left(m\frac{\vec{k}_H}{k_H} - k_H\hat{k}\right)}$$

Show that group velocity is normal to wavenumber vector, i. e., $\vec{c}_g \cdot \vec{K} = 0$:

$$\vec{c}_g \cdot \vec{K} = \frac{Nm}{|\vec{K}|^3}\left(m\frac{\vec{k}_H}{k_H} - k_H\hat{k}\right)\cdot\left(\vec{k}_H + m\hat{k}\right)$$

$$= \frac{Nm}{|\vec{K}|^3}\left(m\frac{\boxed{\vec{k}_H\cdot\vec{k}_H}^{-->k_H^2}}{k_H} - mk_H\boxed{\hat{k}\cdot\hat{k}}^{-->1}\right)$$

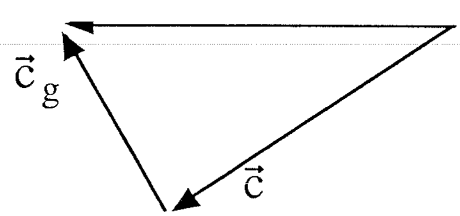
$$= \frac{Nm}{|\vec{K}|^3}\left(mk_H - mk_H\right) = 0$$

"phase velocity" $\vec{c} = \frac{\omega}{|\vec{K}|}\frac{\vec{K}}{|\vec{K}|} = \frac{Nk_H}{|\vec{K}|^3}\left(\vec{k}_H + m\hat{k}\right)$

So $\boxed{\vec{c}_g \cdot \vec{c} = 0}$

Also, vertical part of $\vec{c}$ is exactly opposite that of $\vec{c}_g$, so $\vec{c} + \vec{c}_g$ is always horizontal.

And also, horizontal parts of $\vec{c}$ and $\vec{c}_g$, while not necessarily equal to each other, are in same dir^n (as $\vec{k}_H$).



Examine Figs 7.35, 7.36 from Kundu.

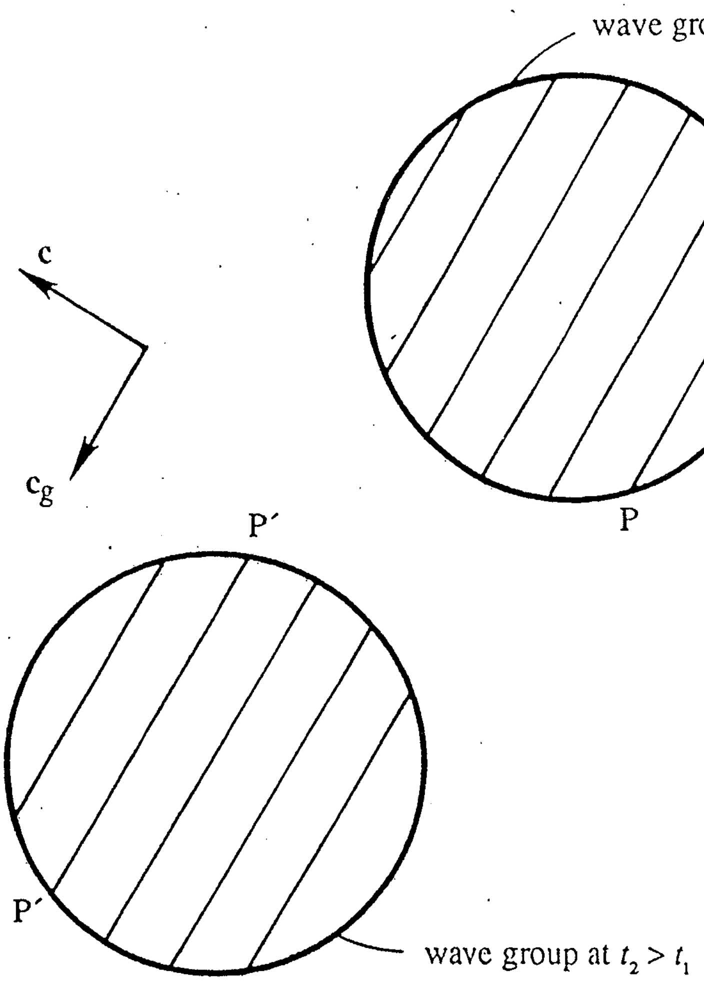


Fig. 7.35 Illustration of phase and group propagation in internal waves. Positions of a wave group at two times are shown. The phase line PP at time t_1 propagates to P′P′ at t_2.

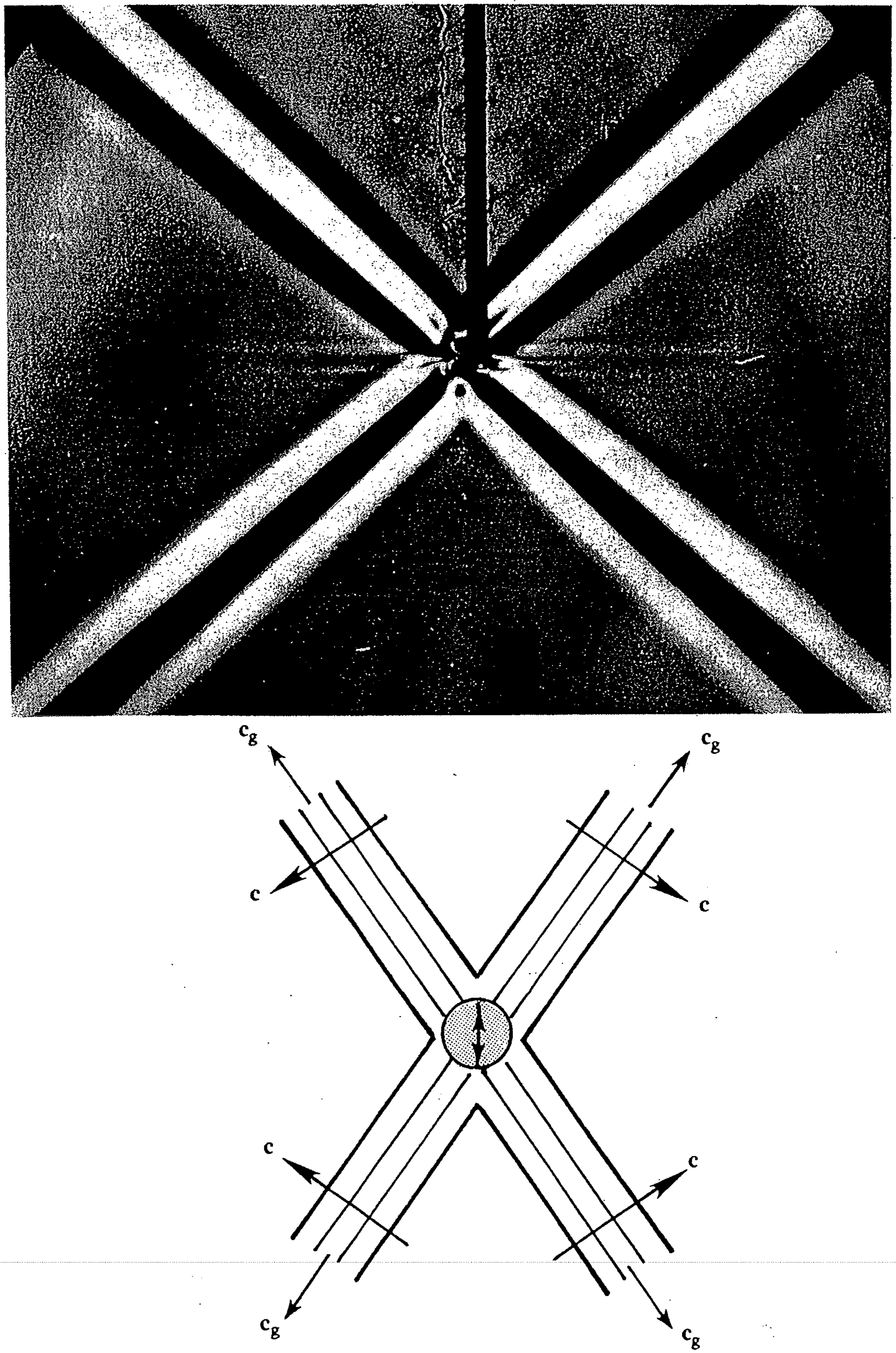


Fig. 7.36 Waves generated in a stratified fluid of uniform buoyancy frequency $N = 1$ rad/s. The forcing agency is a horizontal cylinder, with its axis perpendicular to the plane of the paper, oscillating vertically at frequency $\omega = 0.71$ rad/s. With $\omega / N = 0.71 = \cos\theta$, this agrees with the observed angle of $\theta = 45°$ made by the beams with the horizontal. The vertical dark line in the upper half of the photograph is the cylinder support and should be ignored. The light and dark radial lines represent contours of constant ρ' and are therefore constant phase lines. The schematic diagram below the photograph shows the directions of **c** and $\mathbf{c}_g$ for the four beams. [Photograph supplied by Dr. T. N. Stevenson, University of Manchester.]