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An important thermodynamic derivation

First law of thermodynamics:

$$dQ = dU + dw$$

Change in heat = Change in internal energy + work done

$$dQ = du + dw$$

$$dQ = c_v dT + p\,d\alpha$$

$$dQ = (c_p - R)\,dT + R\,dT - \alpha\,dp$$

$$\boxed{dQ = c_p\,dT - \alpha\,dp}$$

If process is adiabatic,

$dQ \equiv 0$.

$\therefore \quad c_p\,dT = \alpha\,dp.$

Let's examine change in height, dz.

$$c_p \frac{dT}{dz} = \alpha \frac{dp}{dz}.$$

Hydrostatic relation:

$$\frac{dp}{dz} = -\rho g$$

$$c_p \frac{dT}{dz} = \alpha(-\rho g)$$

$$** \quad \boxed{\frac{dT}{dz} = -\frac{g}{c_p} = -9.8\,\frac{K}{km}}$$

"scratch paper"

Pressure = $\frac{\text{Force}}{\text{area}}$, $p = \frac{F}{A}$

$\therefore \; F = pA$

Work = Force × distance

$W = F \times l$

$= pA \times l$

$dw = pA \times dl$; $\rho = \frac{m}{V}$

$dw = p\,dV$ $\quad \alpha \equiv 1/\rho$

per unit mass: $\boxed{dw = p\,d\alpha}$ $\quad \therefore \alpha = \frac{V}{m}$

Change in heat = c_v × Change in Temp, at constant volume & with no work done.

$\therefore \; dQ = c_v\,dT = dU$

$\boxed{dU = c_v\,dT}$

Equation of State: $p = \rho R T$

$p\alpha = RT$

$d(p\alpha) = d(RT)$

$\alpha\,dp + p\,d\alpha = R\,dT$

$\boxed{p\,d\alpha = R\,dT - \alpha\,dp}$

At constant pressure:

$p\,d\alpha = R\,dT - \cancel{\alpha\,dp}$

$\boxed{\frac{p\,d\alpha}{dT} = R}$

$dq = du + dw$

$c_p\,dT = c_v\,dT + p\,d\alpha$

$c_p = c_v + p\frac{d\alpha}{dT}$

$\boxed{c_p = c_v + R}$

②

Now, consider a pseudo-adiabatic process :

$$dq \neq 0 = -L_v\, dw_o$$

L_v : latent heat of vaporization

w_o = amount of water vapor condensed (mixing ratio)

First law of thermodynamics becomes

$$-L_v\, dw_o = C_p\, dT - \alpha\, dp$$

$$-L_v \frac{dw_o}{dz} = C_p \frac{dT}{dz} - \alpha \frac{dp}{dz}$$

Hydrostatic assumption:

$$\frac{dp}{dz} = -\rho g$$

$$-L_v \frac{dw_o}{dz} = C_p \frac{dT}{dz} + g$$

$$w_o \simeq \frac{\varepsilon e_s(T)}{p}, \quad \varepsilon = \frac{R_d}{R_v} = 0.622$$

$= \frac{\text{gas constant dry}}{\text{gas constant wet}}$

$$\frac{dw_o}{dz} = -\frac{C_p}{L_v}\frac{dT}{dz} - \frac{g}{L_v}$$

$$\frac{1}{w_o}\frac{dw_o}{dz} = \underbrace{\frac{1}{e_s}\frac{de_s}{dT}}_{\text{Clausius-Clapeyron}}\frac{dT}{dz} \underbrace{- \frac{1}{p}\frac{dp}{dz}}_{-g/RT}$$

$$\therefore \frac{de_s}{dT} = \frac{L_v e_s}{R_v T^2}$$

$$\therefore \frac{dw_o}{dz} = \frac{L_v w_o}{R_v T^2}\frac{dT}{dz} + \frac{g\, w_o}{RT}$$

$$\therefore \frac{L_v w_o}{R_v T^2}\frac{dT}{dz} + \frac{g\, w_o}{RT} = -\frac{C_p}{L_v}\frac{dT}{dz} - \frac{g}{L_v}$$

after rearranging and some algebra,

$$\boxed{-\frac{dT}{dz} \equiv \frac{g}{c_p}\frac{\left(1 + \frac{L_v w_o}{RT}\right)}{\left(1 + \frac{\varepsilon L_v^2 w_o}{c_p T^2 R}\right)}}$$

= pseudo-adiabatic lapse rate.

* depend on T (and indirectly P)

$$\neq -9.8 \frac{K}{Km} \; !$$

$$\simeq -6.0 \frac{K}{Km}.$$