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ANALYTICAL AND NUMERICAL STUDY OF NATURAL CONVECTION IN A STABLY STRATIFIED FLUID ALONG VERTICAL PLATES AND CYLINDERS WITH TEMPORALLY-PERIODIC SURFACE TEMPERATURE VARIATIONS

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ABSTRACT

This paper describes one-dimensional (parallel) laminar and transitional regimes of natural convection in a viscous stably stratified fluid due to temporally-periodic variations in the surface temperature of infinite vertical plates and cylinders. Analytical solutions are obtained for the periodic laminar regime for arbitrary values of stratification, Prandtl number and forcing frequency.

The solutions for plates and cylinders are qualitatively similar and show that (***i***) the flows are composed of two waves that decay exponentially with distance from the surface; a fast long wave and a slow short wave, (***ii***) for forcing frequencies less than the natural frequency, both waves propagate away from the surface, while (***iii***) for forcing frequencies less than this natural frequency, the short wave propagates away from the surface while the long wave propagates toward the surface.

The analytical results are complemented, for the plate problem, with three-dimensional numerical simulations of flows that start from rest and are suddenly subjected to a periodic thermal forcing. The numerical results depict the transient (start-up) stage of the laminar flow and the approach to the periodicity, and confirm that the analytical solutions provide the appropriate description of the periodic regime for the laminar convection case. Preliminary numerical data are presented for transition from the laminar to turbulent convection.

NOMENCLATURE

k – nondimensional wavenumber;
R – nondimensional radial coordinate;
T' –perturbation temperature;
T_∞ – ambient temperature;
W – nondimensional velocity along the plate;
β – buoyancy parameter;
θ – nondimensional temperature perturbation;
κ – molecular thermal diffusivity;
ξ – nondimensional plate-normal coordinate;
τ – nondimensional time.

INTRODUCTION

Unsteady natural convection flows abound in nature and technology. Such flows are notoriously difficult to analyze theoretically because of the intrinsic coupling between the temperature and velocity fields.

A notable exception is the classical problem of unsteady laminar one-dimensional natural convection along an infinite vertical plate, a class of flows for which the Boussinesq equations of motion and thermodynamic energy reduce to a set of linear partial differential equations that may be solved analytically in a number of circumstances [1]. These solutions are among the few known exact solutions of unsteady natural convection. Such solutions are prized for the insights they provide into flow behavior and for their utility as benchmark solutions for verification of computational fluid dynamics algorithms.

In the 1950's and 1960's, analytical solutions for unsteady one-dimensional natural convection along an infinite vertical plate in an unstratified fluid were obtained for a variety of surface forcings [1].

These solutions described resting fluids that were abruptly set into motion through the agency of a surface heat flux or temperature perturbation. The flows were characterized by a sudden burst of convection along the plate followed by an inexorable outward growth of the boundary layer.

The extension of the one-dimensional convection framework to include ambient stratification is a relatively recent development [2-5].

Shapiro and Fedorovich [4] considered convection in a stratified flow adjacent to a single infinite plate. Analytical solutions were obtained for a Prandtl number of unity for the cases of impulsive (step) change in plate perturbation temperature, sudden application of a plate heat flux, and for arbitrary temporal variations in plate perturbation temperature or plate heat flux. Vertical motion in a stably stratified fluid was associated with a simple negative feedback mechanism: rising warm fluid cooled relative to the environment, whereas subsiding cool fluid warmed relative to the environment. Because of this feedback, convective flows in stably stratified fluid adjacent to a double-infinite plate eventually approached a steady state, whereas the corresponding flows in an unstratified fluid did not.

In a companion paper, Shapiro and Fedorovich [5] explored the Prandtl number dependence of convection of a stably stratified fluid along a single vertical plate both numerically and analytically. The developing boundary layers were thicker, more vigorous, and more sensitive to the Prandtl number at smaller Prandtl numbers (<1) than at larger Prandtl numbers (>1). The gross temporal behavior of the flow after the onset of convection was of oscillatory-decay type for Prandtl numbers near unity, and of non-oscillatory-decay type for large Prandtl numbers.

In the present paper, analytical solutions are obtained for laminar natural convection flows of linearly-stratified fluid along infinite vertical plates and cylinders with temporally-periodic surface temperature variations. The plate and cylinder solutions, which are rather similar in the periodic regime, are valid for arbitrary ambient stratification, forcing frequency and Prandtl number. However, these solutions only apply to the purely periodic regime, and do not describe the transient (start-up) stage of a flow started from rest.

In order to study this start-up flow stage and verify that the analytical solutions provide the appropriate description of the terminal state of the laminar convective flow started from rest, a direct numerical simulation is invoked. Numerical results are presented also for the plate flow undergoing a transition from the laminar to turbulent regime.

ANALYTICAL SOLUTIONS

The governing equations for one-dimensional (parallel) laminar natural convection in the Boussinesq approximation are discussed in detail in Shapiro and Fedorovich [4]. Under the one-dimensional restriction, the Boussinesq form of mass conservation is satisfied identically, while the horizontal equations of motion reduce to statements that the horizontal components of the pressure gradient are zero. The dimensionless vertical equation of motion and thermodynamic energy equation for a linearly-stratified fluid reduce to

$$\frac{\partial W}{\partial \tau} = \theta + \frac{\partial^2 W}{\partial \xi^2}, \quad \frac{\partial \theta}{\partial \tau} = -W + \frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial \xi^2}, \quad (1)$$

where ξ is the horizontal (plate-normal) coordinate, τ is time, θ is perturbation temperature, W is vertical velocity, and Pr=ν/κ is the Prandtl number, where the kinematic viscosity ν and thermal diffusivity κ are considered to be constant. These nondimensional variables (ξ, τ, θ, W) are related to their dimensional counterparts (x, t, T', w) by $\xi \equiv x\nu^{-1/2}(\gamma\beta)^{1/4}$, $\tau \equiv t(\gamma\beta)^{1/2}$, $\theta \equiv T'/T'_0$, $W \equiv w\gamma^{1/2}\beta^{-1/2}/T'_0$, where T'_0 is the surface temperature perturbation, $\gamma \equiv dT_\infty/dz + g/c_p$ is the stratification parameter, z is height, and $T_\infty(z)$ is a linearly-varying ambient temperature.

The corresponding inviscid system [dimensional version of (1) with ν=κ=0] admits traveling wave solutions of the form $w \propto \sin(kx - \sqrt{\beta\gamma}\tau)$ and $T' \propto \cos(kx - \sqrt{\beta\gamma}\tau)$, where $\beta \equiv g/T_r$ is the buoyancy parameter (T_r is the reference temperature); and *en masse* temporal oscillations of the form $w \propto \sin(\sqrt{\beta\gamma}\tau)$ and $T' \propto \cos(\sqrt{\beta\gamma}\tau)$. These inviscid solutions have a frequency equal to the Brunt-Väisälä (buoyancy) frequency $\sqrt{\beta\gamma}$. The nondimensional value of this frequency is unity.

The plate is located at ξ=0, and fluid fills the semi-infinite domain ξ>0. The no-slip condition is imposed at the surface, $W(0, \tau) = 0$. The perturbation temperature at the surface is a temporal oscillation with circular frequency ω and an amplitude of unity, $\theta(0, \tau) = \cos\omega\tau$. This fixed amplitude is a consequence of the nondimensionalization, and does not represent a loss of generality. Since the thermal forcing originates at the plate, and the medium is viscous, the disturbance is considered to vanish far from the plate, except for the case of resonance (imposed frequency equal to the natural frequency of the inviscid system, $\omega = 1$), where an extension of the disturbance to infinity is found to be unavoidable.

Solution for the plate flow: We seek solutions of (1) in the form of simple harmonic oscillations:

$$\theta = \Re[A(\xi)\exp(-i\omega\tau)],\ W = \Re[B(\xi)\exp(-i\omega\tau)], \quad (2)$$

where, A and B are complex-valued functions, and, without loss of generality, ω is considered to be positive. Application of (2) in (1) yields

$$-i\omega A = -B + \frac{1}{\Pr}\frac{d^2A}{d\xi^2}, \quad -i\omega B = A + \frac{d^2B}{d\xi^2}. \quad (3)$$

The solution of (3) is

$$\begin{aligned}\theta &= \frac{1}{2}(1+F_\theta)\exp(-k_+\xi)\cos(k_+\xi-\omega\tau)\\ &+\frac{1}{2}(1-F_\theta)\exp(-k_-\xi)\cos(Sk_-\xi-\omega\tau)\end{aligned}, \quad (4)$$

$$\begin{aligned}W &= F_W\exp(-k_+\xi)\sin(k_+\xi-\omega\tau)\\ &- F_W\exp(-k_-\xi)\sin(Sk_-\xi-\omega\tau)\end{aligned}, \quad (5)$$

where

$$F_\theta \equiv \frac{\omega(\Pr-1)}{\sqrt{4\Pr+\omega^2(\Pr-1)^2}},\ F_W \equiv \frac{1}{\sqrt{4\Pr+\omega^2(\Pr-1)^2}},$$

$$k_+ \equiv \frac{1}{2}\sqrt{\omega(\Pr+1)+\sqrt{4\Pr+\omega^2(\Pr-1)^2}},$$

$$k_- \equiv \frac{1}{2}\sqrt{\left|\omega(\Pr+1)-\sqrt{4\Pr+\omega^2(\Pr-1)^2}\right|},$$

and S is the sign function of $\omega-1$:

$$S = \operatorname{sgn}(\omega-1) = \begin{cases}1, & \omega>1\\ -1, & \omega<1\end{cases}.$$

The special case $\omega = 1$ is considered separately below.

For $\omega > 1$ ($S = 1$, $Sk_-\xi > 0$), the solution is composed of two waves that propagate away from the plate and decay exponentially with distance from the plate. Since the phases of these waves are $k_+\xi - \omega\tau$ and $Sk_-\xi - \omega\tau$, we see that k_+ and k_- can be interpreted as wavenumbers. Since $k_+ > k_- > 0$, the k_- wave has a larger wavelength ($2\pi/k_-$), greater phase speed (ω/k_-), and larger e-folding penetration distance ($1/k_-$) than the k_+ wave.

For $\omega < 1$ ($S = -1$, $Sk_-\xi < 0$), the solution is again composed of a k_+ wave and a k_- wave, each of which decays exponentially with distance from the plate, with the k_- wave having a greater wavelength, greater phase speed and greater penetration distance than the k_+ wave. However, in this case, while the k_+ wave still propagates away from the plate, the k_- wave propagates toward the plate. It can be shown that for both $\omega < 1$ and $\omega > 1$, the k_+ and k_- waves are each associated with positive group velocities, that is, energy propagating away from the plate. We interpret the case of $\omega = 1$ as a case of resonance (forcing frequency equal to the buoyancy frequency – the natural frequency of the inviscid system). A solution that vanishes at infinity is not possible in this case. However, we may require that the disturbance is bounded at infinity. The solution for the case $\omega = 1$ is then found to be

$$\begin{aligned}\theta &= \frac{\cos\tau}{1+\Pr} + \frac{\Pr}{1+\Pr}\exp\left(-F_{\Pr}\xi\right)\cos\left(F_{\Pr}\xi-\tau\right)\\ W &= \frac{\sin\tau}{1+\Pr} + \frac{1}{1+\Pr}\exp\left(-F_{\Pr}\xi\right)\sin\left(F_{\Pr}\xi-\tau\right)\end{aligned}, \quad (6)$$

where $F_{\Pr} \equiv \sqrt{(\Pr+1)/2}$. Thus, the resonance solution is comprised of an outward-propagating spatially-decaying wave and an *en masse* inviscid-like temporal oscillation.

Solution for the cylinder flow: In cylindrical coordinates, the governing equations become

$$\begin{aligned}\frac{\partial W}{\partial\tau} &= \theta + \left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}\right)W\\ \frac{\partial\theta}{\partial\tau} &= -W + \frac{1}{\Pr}\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}\right)\theta\end{aligned}, \quad (7)$$

where the same nondimensionalization as in the plate problem is adopted, but with the dimensionless radial coordinate R and the cylinder radius R_c related to their dimensional counterparts r and r_c by $R \equiv r\nu^{-1/2}(\gamma\beta)^{1/4}$, $R_c \equiv r_c\nu^{-1/2}(\gamma\beta)^{1/4}$, respectively. Again seeking solution in the form of simple harmonic oscillations,

$$\theta = \Re[A(R)\exp(-i\omega\tau)],\ W = \Re[B(R)\exp(-i\omega\tau)],$$

we first express $A(R)$ and $B(R)$ in terms of Hankel functions, and then, in order to separate imaginary and real parts, further express them through Kelvin functions as demonstrated in [6]. This yields

$$\begin{aligned}\theta &= \frac{1}{2}(1+F_\theta)F_{N+}\cos\left(-\varphi_{R+}-\omega\tau+\varphi_{Rc+}\right)\\ &+\frac{1}{2}(1-F_\theta)F_{N-}\cos\left(-S\varphi_{R-}-\omega\tau+S\varphi_{Rc-}\right)\end{aligned}, \quad (8)$$

$$\begin{aligned}W &= F_WF_{N+}\sin\left(-\varphi_{R+}-\omega\tau+\varphi_{Rc+}\right)\\ &-F_WF_{N-}\sin\left(-S\varphi_{R-}-\omega\tau+S\varphi_{Rc-}\right)\end{aligned}, \quad (9)$$

where

$$F_{N+} \equiv \frac{N_0\left(k_+ R\sqrt{2}\right)}{N_0\left(k_+ R_c\sqrt{2}\right)},\; F_{N-} \equiv \frac{N_0\left(k_- R\sqrt{2}\right)}{N_0\left(k_- R_c\sqrt{2}\right)},$$

$$\varphi_{R+} \equiv \varphi_0\left(k_+ R\sqrt{2}\right),\; \varphi_{R-} \equiv \varphi_0\left(k_- R\sqrt{2}\right),$$

$$\varphi_{Rc+} \equiv \varphi_0\left(k_+ R_c\sqrt{2}\right),\; \varphi_{Rc-} \equiv \varphi_0\left(k_- R_c\sqrt{2}\right),$$

and N_0, φ_0 are, respectively, the modulus and phase of the Kelvin functions of order zero:

$$\mathrm{ker}_0(y) = N_0(y)\cos\varphi_0(y),\; \mathrm{kei}_0(y) = N_0(y)\sin\varphi_0(y).$$

The wave characteristics in (8) and (9) are similar to those in the plate flow. Since φ_0<0 for all real values of its argument, the k_+ wave propagates away from the cylinder while the k_- wave propagates away from the cylinder if $\omega > 1$ ($S = 1$) but toward the cylinder if $\omega < 1$ ($S = -1$). Since, $\varphi_0(y) \sim -y/\sqrt{2}$ at large y, and such linear dependence is a good approximation of φ_0 throughout much of the range of its argument, the phase of the k_+ wave is

$$-\varphi_{R+} - \omega\tau + \varphi_{Rc+} \simeq k_+(R - R_c) - \omega\tau\,,$$

while the phase of the k_- wave is

$$-S\varphi_{R-} - \omega\tau + S\varphi_{Rc-} \simeq Sk_-(R - R_c) - \omega\tau\,.$$

These approximate phases are the exact phases for the waves in the plate problem, (4) and (5), with distance from the plate ξ replaced by distance from the cylinder $R - R_c$. Equations (8) and (9) apply to cylinders of any size, including wires (cylinders of very small radius). These equations may be evaluated using standard ascending series and asymptotic formulas for Kelvin functions. However, (8) and (9) greatly simplify if we restrict attention to large cylinder radii/small distances from the surface [6].

In the special case of $\omega = 1$, as in the corresponding case of the plate flow, we consider a solution bounded at infinity. The temperature and velocity fields for the case $\omega = 1$ are then found to be

$$\theta = \frac{\cos\tau}{1+\mathrm{Pr}} + \frac{\mathrm{Pr}}{1+\mathrm{Pr}} F_{N+}\cos\left(-\varphi_{R+} - \tau + \varphi_{Rc+}\right), \quad (10)$$

$$W = \frac{\sin\tau}{1+\mathrm{Pr}} + \frac{1}{1+\mathrm{Pr}} F_{N+}\sin\left(-\varphi_{R+} - \tau + \varphi_{Rc+}\right). \quad (11)$$

Again, we find that the resonance solution is comprised of an outward-propagating spatially-decaying wave and an *en masse* temporal oscillation.

NUMERICAL SIMULATION

A numerical model is applied to the flat plate problem for flows starting from rest that are suddenly subjected to a plate perturbation temperature. These numerical results depict the transient (start-up) stage of the flow and, as will be shown below, confirm that the analytical solutions presented in the previous section provide the appropriate description of a later-stage periodic regime. We solve numerically the three-dimensional Boussinesq equations of motion, thermodynamic energy and mass conservation on a staggered Cartesian (x, y, z) grid:

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p'}{\partial x_i} + \beta T'\delta_{i3} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (12)$$

$$\frac{\partial T'}{\partial t} + u_j\frac{\partial T'}{\partial x_j} = -\gamma u_3 + \kappa\frac{\partial^2 T'}{\partial x_j \partial x_j}, \quad (13)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (14)$$

where $p' = (p - p_\infty)/\rho_r$ is the normalized pressure deviation from its hydrostatic value, i=1,2,3; j=1,2,3 $\mathbf{u} = (u_1, u_2, u_3) \equiv (u, v, w)$ is the three-dimensional flow velocity vector with the components along the coordinate axes $x \equiv x_1$, $y \equiv x_2$, and $z \equiv x_3$, and δ_{ij} is the Kronecker delta.

The employed numerical procedure is described in [5], where a validation test of the computer code is also presented. The velocity components and the perturbation temperature are treated as prognostic variables, while the perturbation pressure is diagnosed from a Poisson equation. At the computational boundary far from the plate (large x), a zero gradient condition is imposed. Periodic boundary conditions are imposed for all variables on the four x-y and x-z computational boundaries. The simulation output is appropriately nondimensionalized to facilitate comparisons with the analytical results.

RESULTS AND DISCUSSION

Figure 1 depicts numerical and analytical solutions for θ and W for Pr=1 and $\omega = 0.5$, a case where the forcing frequency is less than the natural frequency. In the region of the main disturbance, near the plate, the numerical solution approaches the analytical one within just a few oscillation periods. Far from the plate, the disturbance becomes weak and the negative slopes of the zero contours in the analytical solutions for θ and W indicate an inward-propagating wave.

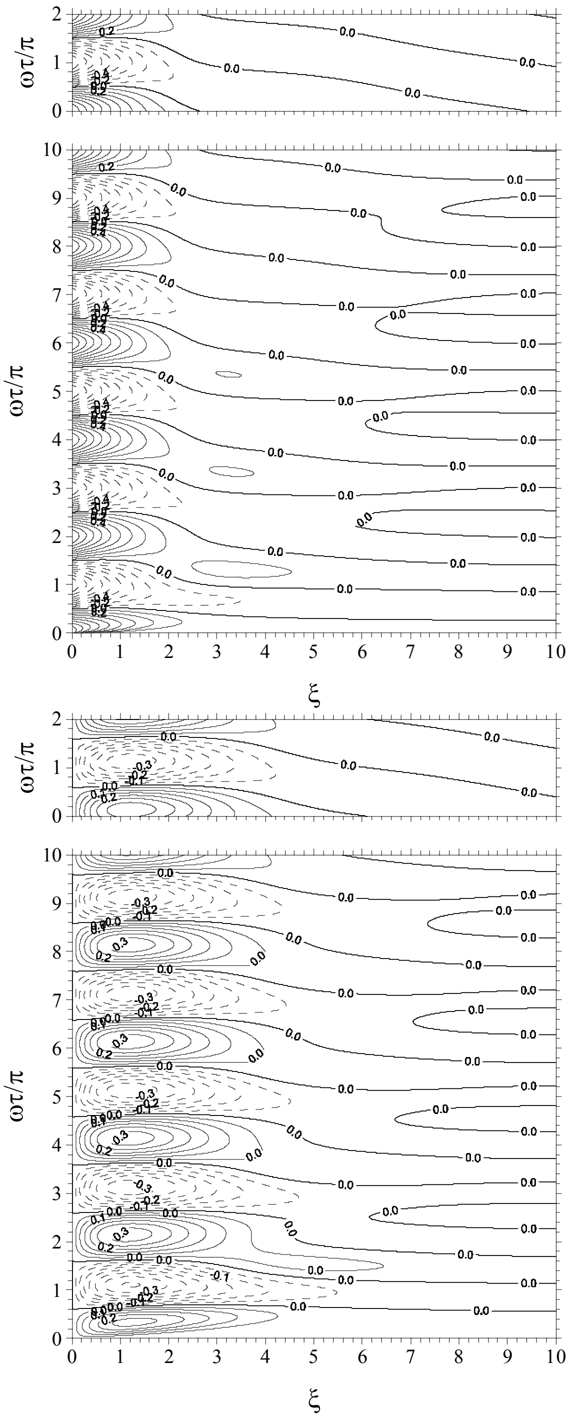

Figure 1.
Plots of θ (upper plot) and W (lower plot) for Pr=1 and ω = 0.5 for the first five oscillation periods. At the top of each figure, corresponding analytical solution for one period of oscillation is presented. Positive (and zero) contours are solid; negative contours are dashed.

Such negative slopes begin to emerge in the numerical solution at the later times in the figure.

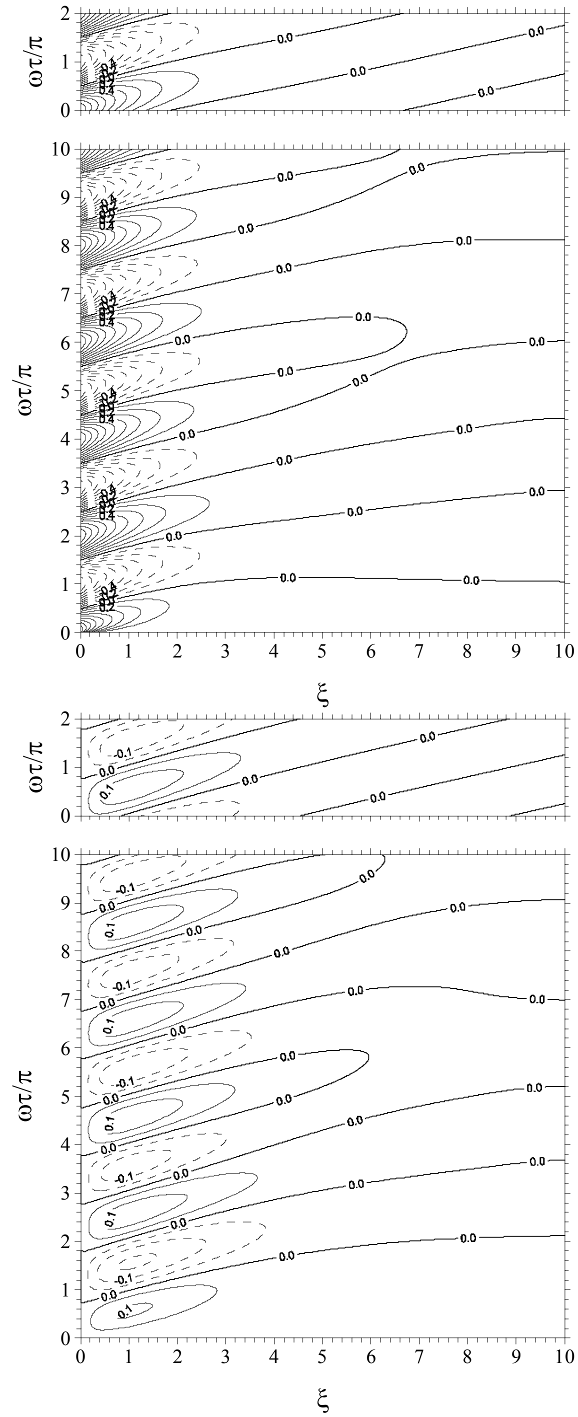

Figure 2.
Plots of θ (upper plot) and W (lower plot) for Pr=1 and ω = 2. Figure details are as in Fig. 1.

Figure 2 presents θ and W solutions for Pr=1 and $\omega = 2$. In this case, the positive slopes of the zero contours for θ and W far from the plate indicate an outward-propagating wave.

The case with Pr=1 and $\omega = 1$ (resonance case), is depicted in Fig. 3. Here, the numerical solution approaches the analytical solution at large ξ relatively slowly. The periodic regime is characterized by *en masse* oscillations of the entire fluid, with the exception of the boundary layer in the immediate vicinity of the plate whose structure is dominated by the no-slip condition.

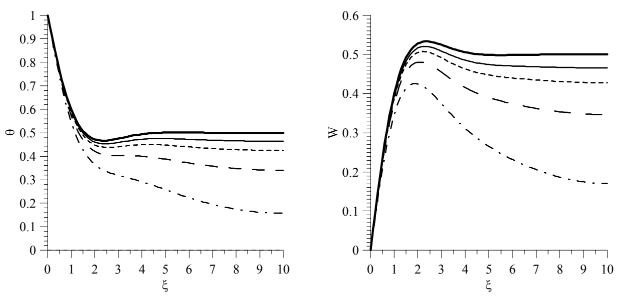

Figure 3.
Numerical solutions for θ (left) and W (right) as functions of ξ for Pr=1 and ω = 1 at different 2π intervals of time. Each W curve is a quarter-period ahead of the corresponding θ curve. The θ curves correspond to τ/π=8 (dashed and dotted line), τ/π=18 (long-dash line), τ/π=28 (short-dash line), and τ/π=38 (solid line). Bold lines show the analytical solutions.

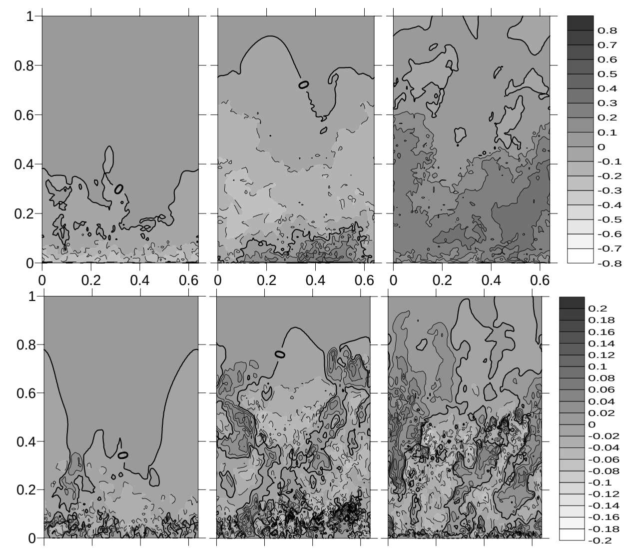

Figure 4.
Along-plate (upper plots) and normal-to-plate (lower plots) velocity components in the turbulent boundary layer (over *x-z* planes) with ω = 1.8 at τ = 1.9, 3.8, and 5.0 (from left to right). Scales are in m s^{-1}.

Figure 4 depicts the development of the turbulent convective boundary layer along the plate in the case Pr=1 and Re$\equiv LV/\nu = T'_0\nu^{-1/2}\gamma^{-3/4}\beta^{1/4}$=7400, where $L \equiv \nu^{1/2}(\gamma\beta)^{-1/4}$ and $V \equiv T'_0\gamma^{-1/2}\beta^{1/2}$, with ω= 1.8 (forcing frequency about twice the natural frequency). In this case, the developing turbulent boundary layer shows clear features of periodicity in the along-plate velocity component that switches from negative to positive values close to the plate, with the flow farther away from the plate lagging in phase behind the near-surface flow. The normal-to-plate velocity field, which is not directly affected by the near-surface temperature oscillations, shows much smaller temporal variability. However, turbulence variances of this component are considerably smaller than those in the boundary layer driven by constant temperature perturbation (not shown) at comparable times.

CONCLUSIONS

The numerical results confirm that the analytical solutions provide appropriate descriptions of impulsively-started oscillatory convection after sufficiently long periods of time. The numerical results are now being extended to the transitional and turbulent regimes. Further results on the laminar and turbulent solutions will be shown at the conference.

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