

Proceedings of 3ICCHMT  
 3<sup>rd</sup> International Conference on Computational Heat and Mass Transfer  
 May 26–30, 2003, Banff, CANADA  
 Paper Number “87” , pp. 606-615

## PRESSURE WORK EFFECTS IN UNSTEADY CONVECTIVELY DRIVEN FLOW ALONG A VERTICAL PLATE

Alan Shapiro and Evgeni Fedorovich  
*School of Meteorology, University of Oklahoma, Norman, OK, USA*

### ABSTRACT

This paper revisits the classical problem of convectively driven one-dimensional (parallel) flow along an infinite vertical plate. We consider flows induced by an impulsive (step) change in plate temperature and by a sudden application of a plate heat flux. Provision is made for pressure work and vertical temperature advection in the thermodynamic energy equation, processes that are generally neglected in previous one-dimensional studies of this problem. In a statically stable environment these additional processes provide a simple negative feedback mechanism: warm air rises, expands and cools relative to the environment, whereas cool air subsides, compresses and warms relative to the environment. Exact solutions of the viscous equations of motion are obtained by the method of Laplace transforms for the case where the Prandtl number is unity. The pressure work and vertical temperature advection are found to have a significant impact on the structure of the solutions at later times.

### NOMENCLATURE

$c_p$  – specific heat at constant pressure ( $=const$ );  
 $g$  – gravitational acceleration ( $\cong 9.81 \text{ m s}^{-2}$ );  
 $p$  – pressure  
 $t$  – time  
 $T$  – temperature ( $T'$  is its deviation from the environmental value)  
 $w$  – velocity component along the plate;  
 $W$  – non-dimensional velocity along the plate;  
 $x$  – direction normal to the plate;  
 $z$  – direction along the plate;  
 $\beta$  – buoyancy parameter;  
 $\kappa$  – molecular thermal diffusivity ( $=const$ );  
 $\theta$  – non-dimensional temperature perturbation;  
 $\nu$  – kinematic viscosity coefficient ( $=const$ );  
 $\tau$  – non-dimensional time;  
 $\xi$  – non-dimensional  $x$  coordinate.

### INTRODUCTION

The transient natural convection flow of a viscous fluid adjacent to vertical surfaces is a fundamental problem in fluid mechanics and heat transfer, with significance for a variety of engineering applications [1]. The simplest form of this problem is one-dimensional transient convective flow adjacent to an infinite vertical plate, first considered in [2] for an impulsive change in plate temperature. Authors of [3]-[6] have obtained analytic solutions to this problem for a variety of temporal variations in plate

temperature and plate heat flux. In these studies pressure work is neglected and ambient thermal stratification is not considered. Accordingly, the thermodynamic energy equation reduces to the standard one-dimensional heat conduction equation. After solving this equation for the temperature field, the vertical velocity is recovered from the vertical equation of motion which has the form of a diffusion equation with inhomogeneous buoyancy forcing term. These exact unsteady solutions of the Boussinesq equations are potentially valuable as simple conceptual/pedagogical models of natural convection as well as tools for validating numerical models of convection.

The present study refines the classical theory of one-dimensional transient convectively driven flow along a vertical plate by including the pressure work term in the thermodynamic energy equation. It also extends the classical theory by making provision for a linearly varying ambient temperature. As we will see, in the context of the one-dimensional model, the pressure work and vertical temperature advection terms are of the same form so the refinement and extension can be taken into account simultaneously by combining both processes into a single advection term. With attention restricted to a perfect gas with a Prandtl number (Pr) of unity, analytical solutions are readily obtained by the method of Laplace transforms. Provision for temperature stratification or pressure work allows the unsteady solution to approach a steady state at large times, whereas if there is no temperature stratification or pressure work (*i.e.*, classical solution), the solution grows without bound. As in the classical case, the new solutions will only be appropriate for times prior to the arrival of the leading edge effect and prior to the onset of any flow instabilities. The steady-state solution for the stratified case has already been obtained in [7], and its linear stability has been analyzed in [8] and [9]. In those studies the stability was found to decrease with increasing plate perturbation temperature and increase with increasing stratification. In light of those studies and the experiments described in [10] and [11], we anticipate that the main interest in our solutions will be in cases where the temperature stratification is large enough to delay (or prevent) flow instability. For weak temperature stratifications, the deviations of our solutions from the corresponding classical solutions may not become apparent before the flow becomes unstable.

The outline of the paper is as follows. We first formulate the problem of one-dimensional natural convection for a fluid, whose thermal expansion coefficient is that of a perfect gas. The governing equations are introduced and reduced to a single fourth order linear partial differential equation for the perturbation temperature. For Pr=1, this equation is solved analytically for the case of an impulsively changed plate perturbation temperature (hereafter referred to as ST case) and for the case of a suddenly applied plate heat flux (hereafter referred to as SF case) by the method of Laplace transforms. The new solutions are then compared to the classical solutions, in which pressure work is neglected and the environment is considered to be isothermal. Lastly, we briefly describe numerical method for computing velocity and temperature solutions for the case of arbitrary Pr value and discuss temperature solutions for different Pr in the case of a suddenly applied plate heat flux.

## GOVERNING EQUATIONS

We consider a Cartesian coordinate system in which the  $z$ -axis opposes the gravity vector, the  $y$ - $z$  plane coincides with an infinite vertical plate, the  $x$ -axis is directed perpendicular to the plate, and fluid fills the region  $x \geq 0$ . The fluid is quiescent with zero horizontal temperature gradient until thermal conditions at the plate are abruptly changed at  $t=0$ . The ensuing motion is one-dimensional with the only non-zero velocity component, the vertical velocity  $w$ , varying only in the  $x$  direction. Accordingly, the mass conservation equation (incompressibility condition) is trivially satisfied. In order for the horizontal equations of motion to be satisfied (albeit trivially), the horizontal pressure gradient force must be zero everywhere. Thus the local pressure  $p(x, z, t)$  must equal  $p_\infty$ , the environmental pressure at  $x \rightarrow \infty$ , which satisfies the hydrostatic equation  $\partial p_\infty / \partial z = -\rho_\infty g$ .

With the density  $\rho(x,z,t)$  and temperature  $T(x,z,t)$  decomposed into its environmental ( $\infty$ ) and perturbation ( $'$ ) components,  $\rho(x,z,t) = \rho_\infty(z) + \rho'(x,t)$ ,  $T(x,z,t) = T_\infty(z) + T'(x,t)$ , and with linearized equation of state,  $\rho'/\rho_r = -T'/T_r$ , the Boussinesq form of the vertical equation of motion is

$$\frac{\partial w}{\partial t} = \beta T' + \nu \frac{\partial^2 w}{\partial x^2}, \quad (1)$$

where the subscript  $r$  denotes a constant reference value, and the term  $g(T'/T_r) = \beta T'$  is the buoyancy force per unit mass of the fluid.

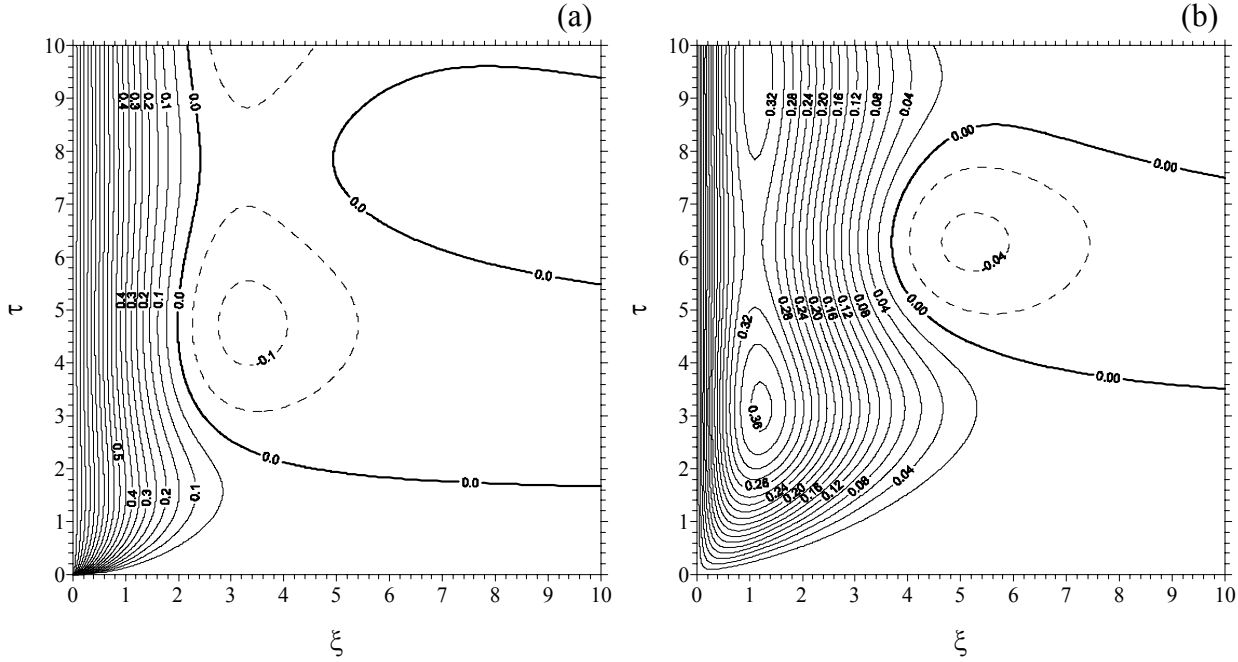


Figure 1. Contours of (a)  $\theta(\xi, \tau)$  and (b)  $W(\xi, \tau)$  for the ST case. The contour increment is 0.02 in  $W(\xi, \tau)$  and 0.05 in  $\theta(\xi, \tau)$ . Negative contours are dashed.

The thermodynamic energy equation in the case under consideration has a form [12]:

$$\rho c_p \left( \frac{\partial T}{\partial t} + w \frac{\partial T}{\partial z} \right) = \frac{\partial p}{\partial t} + w \frac{\partial p}{\partial z} + \rho c_p \kappa \frac{\partial^2 T}{\partial x^2}, \quad (2)$$

where the first two terms on the right-hand side represent the pressure work effect. Decomposing the temperature in the above equation, making use of  $p(x,z,t) = p_\infty$ , restricting attention to linearly varying  $T_\infty(z)$ , and assuming  $\rho/\rho_\infty \approx 1$  we come to

$$\frac{\partial T'}{\partial t} = -\gamma w + \kappa \frac{\partial^2 T'}{\partial x^2}, \quad (3)$$

where  $\gamma = dT_\infty/dz + g/c_p$  is a constant parameter. The  $-\gamma w$  term in (3) arising from the combined effects of pressure work and vertical temperature advection introduces a coupling between  $w$  and  $T'$  beyond the appearance of the buoyancy force in the vertical equation of motion (1). Since the temperature gradient in a statically neutral adiabatic environment is  $dT_\infty/dz = -g/c_p$  [13], we can interpret  $\gamma$  as the difference between the environmental temperature gradient and the temperature gradient in a statically neutral adiabatic environment. Under statically stable conditions ( $\gamma > 0$ ) the  $-\gamma w$  term provides a simple negative feedback in (1) and (3): warm fluid rises, expands and cools relative to the environment, whereas cool

fluid subsides, compresses and warms relative to the environment. Provision for this feedback adds a new level of realism to the classical problem.

The plate boundary conditions for  $t > 0$  are the no-slip condition for velocity,  $w(0,t)=0$ , with either a specified temperature perturbation  $T'|_0$ , as in ST case, or a specified kinematic heat flux  $Q = -\kappa(\partial T'/\partial x)|_0$ , as in SF case, are applied at the plate surface, at  $x=0$ . In both cases  $w$  and  $T'$  are assumed to vanish far from the plate. We introduce dimensionless independent variables  $\xi$  and  $\tau$  as  $\xi \equiv xv^{-1/2}(\gamma\beta)^{1/4}$  and  $\tau \equiv t(\gamma\beta)^{1/2}$ , and scale temperature perturbation  $T'$  and velocity  $w$  depending on the thermal condition specified at the plate. In the ST case, the dependent variables  $w$  and  $T'$  are scaled as  $\theta \equiv T'(T'|_0)^{-1}$  and  $W \equiv w\gamma^{1/2}\beta^{-1/2}(T'|_0)^{-1}$ . In the SF case, the scaling of  $T'$  and  $w$  has the form:  $\theta \equiv T'\kappa(\gamma\beta)^{1/4}v^{-1/2}Q^{-1}$  and  $W \equiv w\kappa\gamma^{3/4}\beta^{-1/4}v^{-1/2}Q^{-1}$ . In both cases, the system (1), (3) becomes:

$$\frac{\partial W}{\partial \tau} = \theta + \frac{\partial^2 W}{\partial \xi^2}, \quad \frac{\partial \theta}{\partial \tau} = -W + \frac{1}{\text{Pr}} \frac{\partial^2 \theta}{\partial \xi^2}, \quad (4)$$

with  $\text{Pr} \equiv \nu/\kappa$  being the Prandtl number. The following boundary conditions for the dimensionless variables  $W(0,\tau)=0$ ,  $W(\infty,\tau)=0$ ,  $\theta(\infty,\tau)=0$ , and either  $\theta(0,\tau)=1$  (ST case) or  $\partial\theta/\partial\xi(0,\tau)=-1$  (SF case). Next, we substitute  $W$  from the second equation of (4) into the first one and obtain the fourth order equation for dimensionless temperature

$$\frac{\partial^2 \theta}{\partial \tau^2} + \frac{1}{\text{Pr}} \frac{\partial^4 \theta}{\partial \xi^4} - (1+1/\text{Pr}) \frac{\partial^3 \theta}{\partial \xi^2 \partial \tau} + \theta = 0, \quad (5)$$

which we will solve analytically in the following sections for the special case of  $\text{Pr}=1$ .

### IMPULSIVE CHANGE IN PLATE TEMPERATURE (ST CASE)

Multiplying (5) by  $e^{-s\tau}$  and integrating (by parts, where possible) from  $\tau=0$  to  $\tau=\infty$  yields the ordinary differential equation

$$\frac{1}{\text{Pr}} \frac{d^4 \hat{\theta}}{d\xi^4} - s(1+1/\text{Pr}) \frac{d^2 \hat{\theta}}{d\xi^2} + (s^2+1)\hat{\theta} = 0, \quad (6)$$

where  $\hat{\theta} \equiv \int_0^\infty \theta e^{-s\tau} d\tau$  is the Laplace transform of  $\theta$ . Restricting attention to  $\text{Pr}=1$ , equation (6) reduces to

$$\frac{d^4 \hat{\theta}}{d\xi^4} - 2s \frac{d^2 \hat{\theta}}{d\xi^2} + (s^2+1)\hat{\theta} = 0, \quad (7)$$

A general solution of (7) for  $\hat{\theta}$  vanishing at  $\xi \rightarrow \infty$  is

$$\hat{\theta} = a \exp(-\xi\sqrt{s+i}) + b \exp(-\xi\sqrt{s-i}). \quad (8)$$

Evaluating coefficients  $a$  and  $b$  from the boundary conditions at the plate (which provides  $a=b=0.5s^{-1}$ ), and taking the inverse transform of  $\hat{\theta}$ , we arrive at

$$\theta = \frac{1}{2} L^{-1} \left[ \frac{1}{s} \exp(-\xi\sqrt{s+i}) \right] + \frac{1}{2} L^{-1} \left[ \frac{1}{s} \exp(-\xi\sqrt{s-i}) \right], \quad (9)$$

where  $L^{-1}$  denotes the inverse Laplace transform operator. The inverse transforms in (9) can be expressed as [14]:

$$L^{-1} \left[ \frac{1}{s} \exp(-\xi\sqrt{s \pm i}) \right] = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau \frac{1}{\tau'^{3/2}} \exp\left(\mp i\tau' - \frac{\xi^2}{4\tau'}\right) d\tau', \quad (10)$$

and so (9) becomes

$$\theta(\xi, \tau) = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau \frac{\cos \tau'}{\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) d\tau'. \quad (11)$$

The velocity field  $W$  is calculated from the second equation of (4) with  $\theta$  supplied from (11). After integration by parts and rearrangement of terms, we obtain

$$W(\xi, \tau) = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau \frac{\sin \tau'}{\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) d\tau'. \quad (12)$$

Although it is not obvious, careful analysis of (11) shows that it provides the desired temperature value at the plate surface:  $\lim_{\xi \rightarrow 0} \theta(\xi, \tau) = 1$ . Differentiation of (11) with respect to  $\xi$  yields

$$-\frac{d\theta}{d\xi}(0, \tau) = \frac{\cos \tau}{\sqrt{\tau\pi}} + \frac{1}{\sqrt{2}}. \quad (13)$$

Thus, the plate heat flux is infinite at  $\tau=0$  and undergoes a decaying oscillation as it approaches  $1/\sqrt{2}$  in the limit  $\tau \rightarrow \infty$ . Since the non-dimensional period  $2\pi$  corresponds to a dimensional period of  $2\pi(\beta\gamma)^{-1/2}$ , stronger stratifications, or larger buoyancy parameter values, are associated with higher oscillation frequencies.

Contour plots of  $\theta(\xi, \tau)$  and  $W(\xi, \tau)$  are presented in Fig. 1. These plots show the boundary-layer character of the solutions and the oscillatory approach to steady-state conditions. The plots were constructed by numerically evaluating the integrals in (11) and (12).

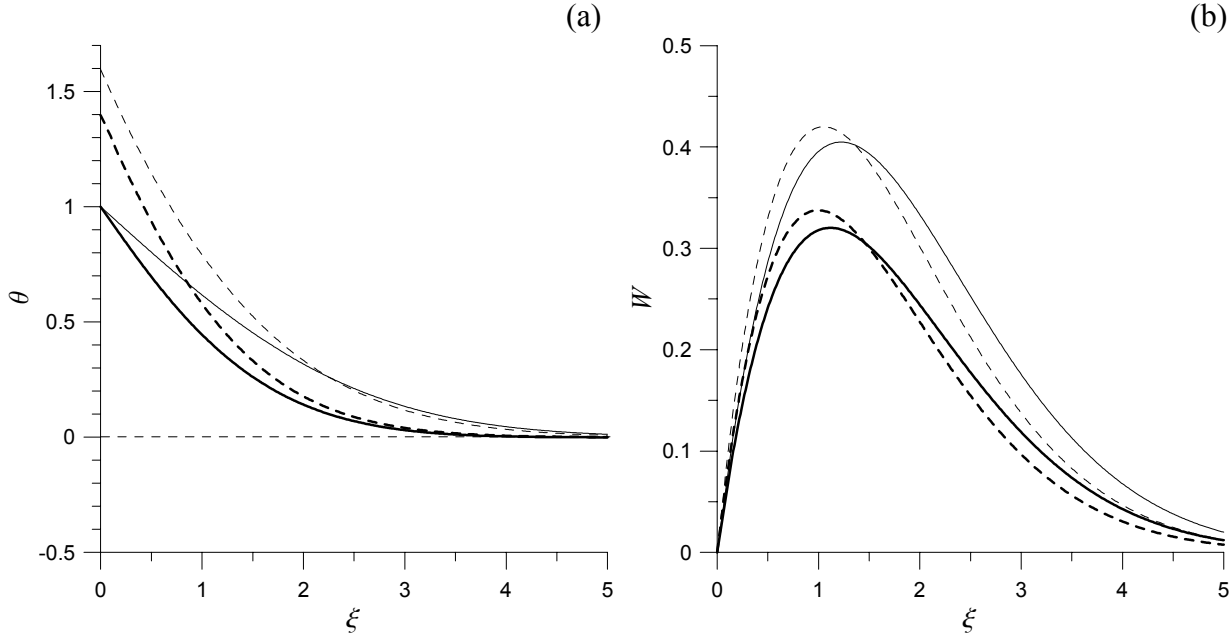


Figure 2. Cross-sections of (a)  $\theta(\xi, \tau)$  and (b)  $W(\xi, \tau)$  at dimensionless time  $\tau = 2$ . Solid line presents classical solution and heavy solid line presents new solution for the case of an impulsive (step) change in plate perturbation temperature. Dashed line presents classical solution and heavy dashed line presents new solution for the case of suddenly applied plate heat flux.

Classical solutions for the ST case when the pressure work is neglected, the environment is considered to be isothermal, and  $Pr=1$ , have the form [5]:

$$\theta(\xi, \tau) = \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right) = \frac{\xi}{2\sqrt{\pi}} \int_0^{\tau} \frac{1}{\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) d\tau', \quad (14)$$

$$W(\xi, \tau) = \xi\sqrt{\tau} \cdot \operatorname{ierfc}\left(\frac{\xi}{2\sqrt{\tau}}\right) = -\frac{\xi^3}{4\sqrt{\pi}} \int_0^{\tau} \frac{1}{\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) d\tau' + \frac{\xi\sqrt{\tau}}{\sqrt{\pi}} \exp\left(-\frac{\xi^2}{4\tau}\right), \quad (15)$$

where  $\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-x'^2) dx'$  is the complementary error function and  $\operatorname{ierfc}(x) \equiv \int_x^{\infty} \operatorname{erfc}(x') dx'$ , see

[15]. To facilitate the comparison, we have non-dimensionalized the classical solutions in the same manner as our new solutions. New and classical solutions for the horizontal distributions of  $\theta(\xi, \tau)$  and  $W(\xi, \tau)$  at  $\tau=2$  in the ST case are compared in Fig. 2. It is clearly seen that the classical model predicts larger temperature perturbations and larger velocities than predicted by the new model that accounts for negative feedback mechanism associated with stratification and pressure work effects.

The steady-state solution for dimensionless temperature  $\theta_s(\xi)$ , corresponding to arbitrary values of Pr, is obtained from (5) with time derivatives neglected. In this case, (5) reduces to a linear fourth-order ordinary differential equation, whose solution in our case is given by

$$\theta_s(\xi) = \cos(\xi \operatorname{Pr}^{1/4} / \sqrt{2}) \exp(-\xi \operatorname{Pr}^{1/4} / \sqrt{2}). \quad (16)$$

The steady-state velocity  $W_s(\xi)$  is readily obtained from the steady version of the second equation of (4):

$$W_s(\xi) = \operatorname{Pr}^{-1/2} \sin(\xi \operatorname{Pr}^{1/4} / \sqrt{2}) \exp(-\xi \operatorname{Pr}^{1/4} / \sqrt{2}). \quad (17)$$

Equations (16) and (17) were first obtained in [7]. It may be shown, see [16], that our unsteady solutions (11) and (12) approach the steady-state solutions (16) and (17) with  $\operatorname{Pr}=1$ .

### SUDDEN APPLICATION OF PLATE HEAT FLUX (SF CASE)

Now we consider the case when a plate heat flux is suddenly applied at  $\tau=0$  (SF case). As in the ST case, we first come to (5) and, after the assumption is made that  $\operatorname{Pr}=1$ , obtain the general expression (8) for the transformed temperature  $\hat{\theta}$ . Applying boundary conditions for the SF case, we come to

$$\hat{\theta} = \frac{1}{2i} (\sqrt{s+i} - \sqrt{s-i}) \left[ \frac{1}{s} \exp(-\xi\sqrt{s+i}) + \frac{1}{s} \exp(-\xi\sqrt{s-i}) \right]. \quad (18)$$

The inverse transform of (16) given by  $\theta = L^{-1}\hat{\theta}$  is evaluated with the convolution theorem used in conjunction with (10):

$$\theta(\xi, \tau) = \frac{\xi}{2\pi} \int_0^{\tau} \frac{\sin(\tau - \tau')}{(\tau - \tau')^{3/2}} \int_0^{\tau'} \frac{\cos \tau''}{\tau''^{3/2}} \exp\left(-\frac{\xi^2}{4\tau''}\right) d\tau'' d\tau'. \quad (19)$$

Comparing the solution (19) for the temperature in the SF case with the temperature solution (11) for the ST case (we will denote it by  $\theta_{\text{ST}}$ ), we have

$$\theta(\xi, \tau) = \frac{1}{\sqrt{\pi}} \int_0^{\tau} \frac{\sin(\tau - \tau')}{(\tau - \tau')^{3/2}} \theta_{\text{ST}}(\xi, \tau') d\tau', \quad (20)$$

which means that at any location  $\xi$  the solution for the SF case is a weighted average over time of the solution for the ST case. To obtain the temperature perturbation at the plate in the SF case, we take a  $\xi \rightarrow 0$  limit of (20). Integration by parts yields

$$\theta(0, \tau) = -\frac{2 \sin \tau}{\sqrt{\tau\pi}} + 2^{3/2} C_F(\sqrt{2\tau/\pi}), \quad (21)$$

where  $C_F(x) \equiv \int_0^x \cos(\pi x'^2/2) dx'$  is a Fresnel cosine integral [15], which tends to  $1/2$  as  $\tau \rightarrow \infty$ . Thus, the

dimensionless plate temperature approaches  $\sqrt{2}$  as  $\tau \rightarrow \infty$ .

The velocity  $W(\xi, \tau)$  is again obtained as residual of the second equation of (4):

$$W(\xi, \tau) = \frac{\xi}{2\pi} \int_0^\tau \frac{\sin(\tau - \tau')}{(\tau - \tau')^{3/2}} \int_0^{\tau'} \frac{\sin \tau''}{\tau''^{3/2}} \exp\left(-\frac{\xi^2}{4\tau''}\right) d\tau'' d\tau'. \quad (22)$$

Comparing (22) and (12), one may see that  $W(\xi, \tau)$  for the SF case can be expressed as weighted average of the  $W(\xi, \tau)$  solution for the ST case (we will denote it by  $W_{ST}$ ):

$$W(\xi, \tau) = \frac{1}{\sqrt{\pi}} \int_0^\tau \frac{\sin(\tau - \tau')}{(\tau - \tau')^{3/2}} W_{ST}(\xi, \tau') d\tau'. \quad (23)$$

Contour plots of  $\theta(\xi, \tau)$  and  $W(\xi, \tau)$  for the SF case obtained by numerical evaluation of the integrals in (19) and (22) are presented in Fig. 3. Equation (21) was used to evaluate  $\theta(0, \tau)$ . The solutions for the SF case are qualitatively similar to the solutions for the ST case (Fig. 1).

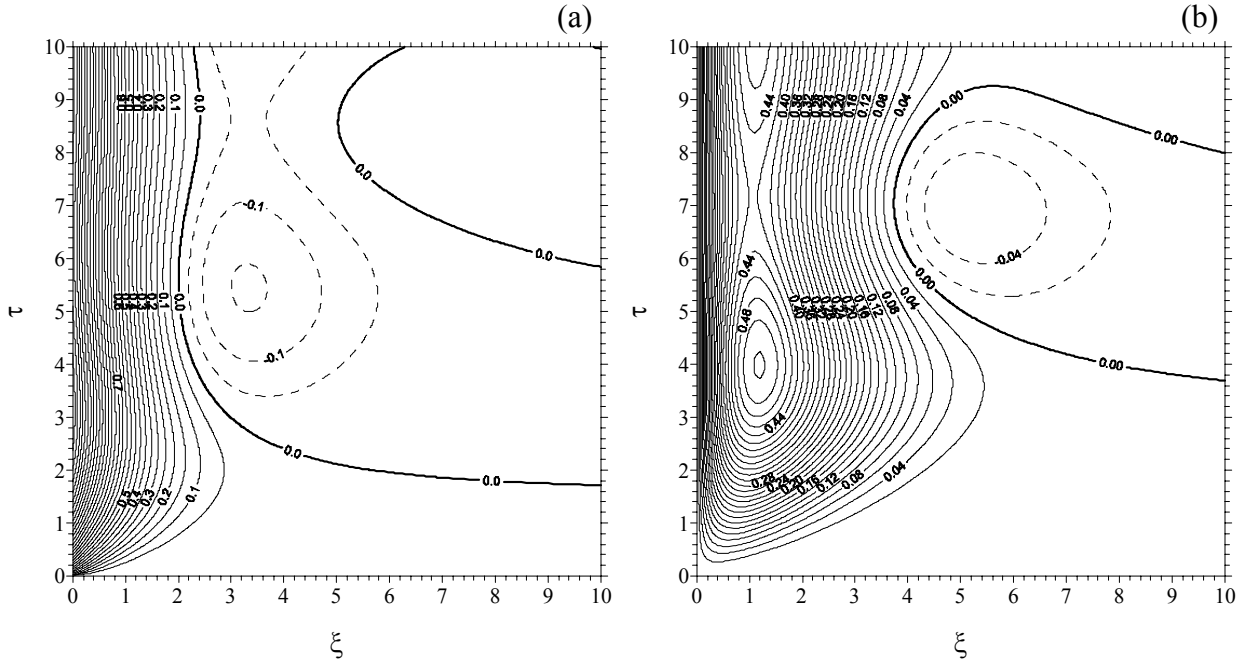


Figure 3. Contours of (a)  $\theta(\xi, \tau)$  and (b)  $W(\xi, \tau)$  for the SF case. Contour increments are as in Fig. 1.

Classical solutions for the SF case when the pressure work is neglected, the environment is considered to be isothermal, and  $Pr=1$ , have the form [5]:

$$\theta(\xi, \tau) = 2\sqrt{\tau} \cdot ierfc\left(\frac{\xi}{2\sqrt{\tau}}\right) = -\frac{\xi^2}{2\sqrt{\pi}} \int_0^\tau \frac{1}{\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) d\tau' + \frac{\xi\sqrt{\tau}}{\sqrt{\pi}} \exp\left(-\frac{\xi^2}{4\tau}\right), \quad (24)$$

$$W(\xi, \tau) = 2\xi\tau \cdot i^2erfc\left(\frac{\xi}{2\sqrt{\tau}}\right) = \frac{\xi^2}{4\sqrt{\pi}} \left(\frac{\xi^2}{2} + \tau\right) \int_0^\tau \frac{1}{\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) d\tau' - \frac{\xi^2\sqrt{\tau}}{2\sqrt{\pi}} \exp\left(-\frac{\xi^2}{4\tau}\right), \quad (25)$$

where  $i^2erfc(x) \equiv \int_x^\infty ierfc(x') dx'$ . Careful analysis of (24) and (25) in comparison with new solutions (19)

and (22) indicates that, analogously to the ST case, the new and classical solutions for the SF case are in close agreement only for  $\tau < 1$ . Already at  $\tau = 2$ , the new solutions for the SF case significantly diverge

from their classical counterparts, see Fig. 2.

The steady-state solutions  $\theta_s(\xi)$  and  $W_s(\xi)$  for the SF case,

$$\theta_s(\xi) = \sqrt{2} \text{Pr}^{-1/4} \cos(\xi \text{Pr}^{1/4}/\sqrt{2}) \exp(-\xi \text{Pr}^{1/4}/\sqrt{2}), \quad (26)$$

$$W_s(\xi) = \sqrt{2} \text{Pr}^{-3/4} \sin(\xi \text{Pr}^{1/4}/\sqrt{2}) \exp(-\xi \text{Pr}^{1/4}/\sqrt{2}), \quad (27)$$

are obtained in the same manner as for the ST case (see previous section). The solutions (26) and (27) differ from their counterparts for the ST case, equations (16) and (17), by a multiplicative factor of  $\sqrt{2} \text{Pr}^{-1/4}$ . As shown in [16], the unsteady solutions (19) and (22) at  $\tau \rightarrow \infty$  approach the steady-state solutions (26), (27) with  $\text{Pr}=1$ .

## NUMERICAL INVESTIGATION OF CONVECTION WITH ARBITRARY Pr NUMBER

The analytical solutions for temperature and vertical velocity discussed in the previous sections of the present paper have been obtained for the particular case of  $\text{Pr}=1$ . In order to investigate Prandtl-number variability of dynamic and thermal regime in the considered convectively driven flow, we addressed the problem numerically. The following system of equations in the Boussinesq approximation was used:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p'}{\partial x_i} + \beta T' \delta_{i3} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (28)$$

$$\frac{\partial T'}{\partial t} + u_j \frac{\partial T'}{\partial x_j} = -\gamma u_3 + \kappa \frac{\partial^2 T'}{\partial x_j \partial x_j}, \quad (29)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (30)$$

where  $p' = (p - p_\infty)/\rho_r$  is the normalized pressure deviation from its hydrostatic value,  $i=1,2,3$ ;  $j=1,2,3$   $\mathbf{u} = (u_1, u_2, u_3) \equiv (u, v, w)$  is the three-dimensional flow velocity vector with the components along the coordinate axes  $x \equiv x_1$ ,  $y \equiv x_2$ , and  $z \equiv x_3$ ,  $\delta_{ij}$  is the Kronecker delta, and the Einstein convention of summation over repeated indices is applied. Other notation corresponds to the one introduced in the "Governing Equations" section. The above system can be used for investigation of both the regimes of laminar and turbulent convection along a heated vertical plate. For the present study, the plate thermal forcing (which is either temperature perturbation or heat flux) and physical parameters of the problem ( $\beta$ ,  $\gamma$ ,  $\nu$ , and  $\kappa$ ) have been chosen in a way to ensure the laminar flow regime.

We discretized equations (28)-(30) in a rectangular domain stretched along the  $x$  axis. Equations for the prognostic variables ( $u$ ,  $v$ ,  $w$ ,  $T'$ ) were integrated over time by the leapfrog scheme with a weak filter, and the spatial derivatives were approximated by the second-order finite-difference expressions, as described in [17]. The Poisson equation for the pressure was solved by the Fast Fourier Transform technique over the  $y$ - $z$  planes, and by the tridiagonal factorization method in the  $x$  direction. The no-slip boundary conditions for the velocity components were applied at the surface of the plate, where either the temperature perturbation value was prescribed (in the ST case) or the temperature gradient related to the surface kinematic heat flux  $Q$  as  $\partial T'/\partial x = -Q/\kappa$  (in the SF case). The normalized pressure at the surface of the plate was calculated from the truncated version of the third ( $i=3$ ) equation of motion. At the opposite side of the domain, at large  $x$ , the normal gradients of all prognostic variables were set to zero. Over the  $x$ - $y$  and  $x$ - $z$  computational boundaries of the domain (four in total), the periodic boundary conditions for all computed variables were applied. The output velocity and temperature perturbation values were averaged over the  $y$ - $z$  planes. However, variations of these quantities in the  $y$ - $z$  planes (*i.e.*, parallel to the wall) were found to be negligibly small.

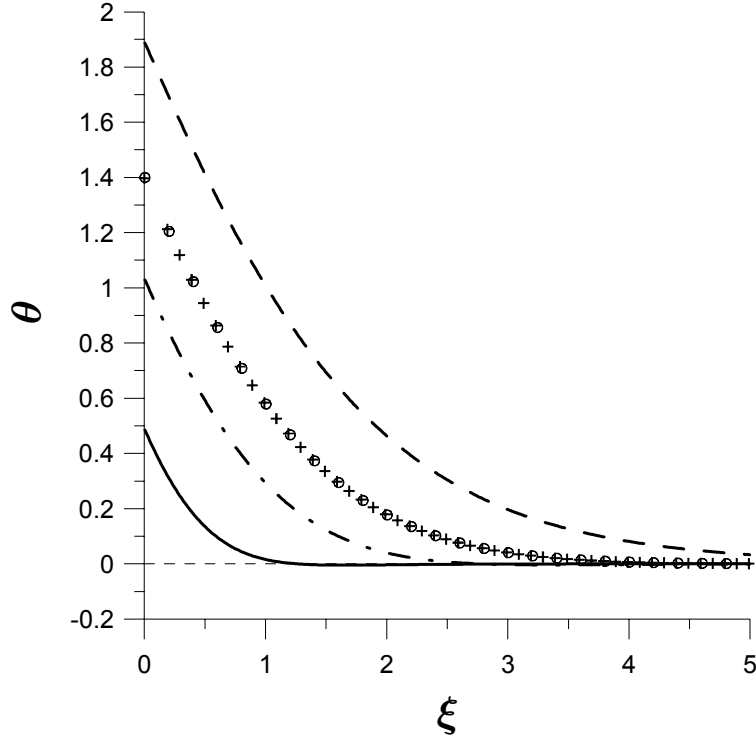


Figure 4. Pr-number dependence of dimensionless temperature  $\theta(\xi, \tau)$  at  $\tau=2$  for the SF case. Solutions for  $Pr=1$  are given by symbols: the crosses present analytical solution and the circles present the numerical solution. Numerical solutions for  $Pr=10$ , 2, and 0.5 are depicted by solid, dashed-dotted, and dashed lines, respectively.

Results of numerical simulations of perturbation temperature in the SF case for four different Pr numbers are presented in Fig. 4. The computed  $\theta(\xi, \tau)$  distribution for  $Pr=1$  almost perfectly overlaps with the corresponding analytical curve (see also Fig. 2), which proves the adequacy of the employed numerical procedure for simulation of the considered flow type. Within the range of investigated Pr number variations (from 0.5 to 10, which covers most of its variability in natural fluids), the dimensionless plate temperature turns out to be rather sensitive to the value of Pr number. Smaller Pr lead to noticeably higher temperature values at the plate and thicker thermal boundary layers.

## CONCLUSIONS

This study revisits one of the simplest scenarios of natural convection, the one-dimensional (parallel) convectively driven flow of a viscous fluid along an infinite vertical plate. Our model refines the classical theory by including the pressure work term in the thermodynamic energy equation, and extends the theory by making provision for a linearly varying ambient temperature. With attention restricted to a Prandtl number of unity, exact analytic solutions of the viscous equations of motion are obtained for flows driven by an impulsively changed plate perturbation temperature, and a suddenly imposed plate heat flux. The considered thermodynamic processes introduce a negative feedback mechanism whereby warm fluid rises, expands and cools relative to the environment, while cool fluid subsides, compresses and warms relative to the environment. This negative feedback mechanism results in a convective flow that approaches a steady state at large times. In contrast, in the classical solutions where pressure work is neglected and there is no temperature stratification, the disturbance continues to spread outward from the plate and no steady state is approached. In these latter flows, the fluid experiences a persistent buoyancy-

induced acceleration, and the vertical velocity grows without bound.

It is straightforward to show, see [16], how the method of Laplace transforms can be used to obtain solutions for arbitrary temporal variations of plate perturbation temperature or plate heat flux, again with a Prandtl number of unity. However, analytic solutions for Prandtl numbers different from unity appear to be more challenging to obtain, and numerical analysis may be the most convenient way to proceed. In this study we have presented preliminary numerical solutions for the case of a suddenly imposed plate heat flux for Prandtl numbers in the range 0.5-10. The numerical results indicate that thicker boundary layers are obtained at smaller Prandtl numbers.

## ACKNOWLEDGEMENTS

The authors gratefully acknowledge the assistance of Robert Conzemius in redesigning the computer code used in the numerical part of this study.

## REFERENCES

1. Gebhart, B., Jaluria, Y., Mahajan, R. L., and Sammakia, B., 1988, *Buoyancy-Induced Flows and Transport*. Hemisphere Publishing.
2. Illingworth, C. R. , 1950, Unsteady Laminar Flow of a Gas Near an Infinite Flat Plate. Proc. Cambridge Philos. Soc., **46**, pp. 603-613.
3. Siegel, R., 1958, Transient Free Convection From a Vertical Flat Plate. Trans. ASME, **80**, pp. 347-359.
4. Menold, E. R., and Yang, K.-T., 1962, Asymptotic Solutions for Unsteady Laminar Free Convection on a Vertical Plate. Trans. ASME: J. Appl. Mech., **29**, pp. 124-126.
5. Goldstein, R. J., and Briggs, D. G., 1964, Transient Free Convection About Vertical Plates and Circular Cylinders. Trans. ASME: J. Heat Transfer, **86**, pp. 490-500.
6. Das, U. N., Deka, R. K., and Soundalgekar, V. M., 1999, Transient Free Convection Flow Past an Infinite Vertical Plate with Periodic Temperature Variation. J. Heat Transfer, **121**, pp. 1091-1094.
7. Gill, A. E., 1966, The Boundary Layer Regime for Convection in a Rectangular Cavity. J. Fluid Mech., **26**, pp. 515-536.
8. Gill, A. E., and Davey, A., 1969, Instabilities of a Buoyancy-Driven System. J. Fluid Mech., **35**, pp. 775-798.
9. Bergholz, R. F., 1978, Instability of Steady Natural Convection in a Vertical Fluid Layer. J. Fluid Mech., **84**, pp. 743-768.
10. Brooker, A. M. H., Patterson, J. C., Graham, T., and Schöpf, W., 2000, Convective Instability in a Time-Dependent Buoyancy Driven Boundary Layer. Intl J. Heat Mass Transfer, **43**, pp. 297-310.
11. Patterson, J. C., Graham, T., Schöpf, W., and Armfield, S. W., 2002, Boundary Layer Development on a Semi-Infinite Suddenly Heated Vertical Plate. J. Fluid Mech., **453**, pp. 39-55.
12. Schlichting, H., 1979, *Boundary-Layer Theory*. 7th edition. McGraw Hill.
13. Holton, J. R., 1992, *An Introduction to Dynamic Meteorology*. Academic Press.
14. Doetsch, G., 1961, *Guide to the Applications of Laplace Transforms*. Van Nostrand.
15. Abramowitz, M., and Stegun, I. A., 1964, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards, Washington, D.C.
16. Shapiro, A., and Fedorovich, E., 2003, Unsteady Convectively Driven Flow Along a Vertical Plate Immersed in a Stably Stratified Fluid. J. Fluid Mech. (in review).
17. Fedorovich, E., Nieuwstadt, F. T. M., and Kaiser, R., 2001, Numerical and Laboratory Study of Horizontally Evolving Convective Boundary Layer. Part I: Transition Regimes and Development of the Mixed Layer. J. Atmos. Sci., **58**, pp. 70-86.