On the Evaluation of the Proportionality Coefficient between the Turbulence Temperature Spectrum and Structure Parameter

JEREMY A. GIBBS

Cooperative Institute for Mesoscale Meteorological Studies, University of Oklahoma, and NOAA/OAR/National Severe Storms Laboratory, Norman, Oklahoma

EVGENI FEDOROVICH

School of Meteorology, University of Oklahoma, Norman, Oklahoma

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ABSTRACT

The turbulence temperature spectrum and structure parameter are related through a widely adopted proportionality coefficient. We formally derive this expression, and present further evidence, to demonstrate that this coefficient is too large by a factor of 2.

1. Introduction

It is customary to quantify the intensity of turbulent fluctuations of meteorological fields through a single parameter called the structure parameter. The structure parameter, which is assumed constant within the inertial subrange of turbulence scales, is relevant to many applications associated with atmospheric boundary layer processes. One of the most studied structure parameters is the (potential) temperature parameter, which is frequently used to describe properties of electromagnetic and acoustic wave propagation in the atmosphere. A relationship between the scalar structure parameter and the scalar spectral density in the inertial subrange was first obtained by Tatarskii (1961) and evaluated for temperature by Wyngaard et al. (1971). The latter form has been widely used in the engineering and atmospheric sciences for nearly fifty years. We formally rederive this relationship and present evidence to demonstrate that the associated proportionality coefficient is too large by a factor of 2.

2. Deriving the integral spectral approximation of the structure parameter

The second-order temperature structure function (Tatarskii 1961; Wyngaard 2010) is defined as

$$D_T(r) = \overline{[T(x) - T(x+r)]^2}, \tag{1}$$

where T is temperature or potential temperature when Eq. (1) is used in the atmospheric context, x is the coordinate direction in space, and $r = |\mathbf{r}|$ is the separation distance (equal to the magnitude of the separation vector). If we assume that turbulence is locally isotropic and that the separation distance lies within the inertial subrange of spatial scales of turbulent temperature fluctuations, the temperature structure function may be expressed as (Kolmogorov 1941)

$$D_T(r) = C_T^2 r^{2/3}, \tag{2}$$

where C_T^2 is the temperature structure-function parameter, often just called the temperature structure parameter. The following relationship between D_T and the one-dimensional spectral density of temperature fluctuations Φ_T is valid under the assumption of turbulence isotropy (Wyngaard 2010):

$$\begin{aligned} D_T &= 2\int_{-\infty}^{\infty} [1-\cos(kr)]\Phi_T(k)\,dk \\ &= 4\int_0^{\infty} [1-\cos(kr)]\Phi_T(k)\,dk, \end{aligned} \tag{3}$$

where k is the wavenumber associated with the x direction. Both Eqs. (2) and (3) are based on the fundamental definition of the structure function and require no additional

Corresponding author: Jeremy A. Gibbs, jeremy.gibbs@noaa.gov

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assumptions beyond their theoretical underpinnings. As an aside, Essenwanger and Reiter [1969, their Eq. (6)]—an oft-cited study focused on structure functions and power spectra of atmospheric velocity increments—incorrectly omits the leading factor of 2 in Eq. (3).

One can evaluate the integral in Eq. (3) analytically by assuming that the entire spectrum has the inertial subrange form

$$\Phi_T = Ak^{-5/3}. \tag{4}$$

To do this, we substitute Eq. (4) into Eq. (3) to obtain

$$D_T = 4A\int_0^\infty [1-\cos(kr)]k^{-5/3}\,dk. \tag{5}$$

We next take the integral employing integration by parts:

$$\int_0^\infty [1-\cos(kr)]k^{-5/3}\,dk = \underbrace{-\frac{3}{2}[1-\cos(kr)]k^{-2/3}\Big|_0^\infty}_{=0} + \frac{3}{2}r\int_0^\infty \sin(kr)k^{-2/3}\,dk. \tag{6}$$

We use the following table integral from Gradshteyn and Ryzhik [2014, their Eq. (3.761), p. 440] for the remaining part on the right-hand side of Eq. (6):

$$\int_0^\infty x^{\mu-1}\sin(ax)\,dx = \frac{\pi}{2a^\mu\Gamma(1-\mu)\cos\left(\frac{\mu\pi}{2}\right)},$$

where $x = k$, $\mu = 1/3$ according to Eq. (6), and $a = r$. This yields

$$\frac{3}{2}r\int_0^\infty \sin(kr)k^{-2/3}\,dk = \frac{3}{2}r\frac{1}{2r^{1/3}\Gamma\left(\frac{2}{3}\right)\cos\left(\frac{\pi}{6}\right)} = \frac{3\pi r^{2/3}}{4\Gamma\left(\frac{2}{3}\right)\cos\left(\frac{\pi}{6}\right)}.$$

Noting that $\cos(\pi/6) = \sin(\pi/3)$ and $\Gamma(2/3) = (3/2)\Gamma(5/3)$, we have

$$\frac{3}{2}r\int_0^\infty \sin(kr)k^{-2/3}\,dk = \frac{3\pi r^{2/3}}{4\frac{3}{2}\Gamma\left(\frac{5}{3}\right)\sin\left(\frac{\pi}{3}\right)} = \frac{\pi r^{2/3}}{2\Gamma\left(\frac{5}{3}\right)\sin\left(\frac{\pi}{3}\right)}.$$

Thus,

$$\int_0^\infty [1-\cos(kr)]k^{-5/3}\,dk = \frac{\pi r^{2/3}}{2\Gamma\left(\frac{5}{3}\right)\sin\left(\frac{\pi}{3}\right)} \tag{7}$$

and

$$D_T = 4A\int_0^\infty [1-\cos(kr)]k^{-5/3}\,dk = \frac{2\pi Ar^{2/3}}{\Gamma\left(\frac{5}{3}\right)\sin\left(\frac{\pi}{3}\right)}. \tag{8}$$

Making use of Eq. (2) gives us

$$\frac{2\pi Ar^{2/3}}{\Gamma\left(\frac{5}{3}\right)\sin\left(\frac{\pi}{3}\right)} = C_T^2 r^{2/3}.$$

Solving for A provides

$$A = \frac{C_T^2\Gamma\left(\frac{5}{3}\right)\sin\left(\frac{\pi}{3}\right)}{2\pi} \approx 0.125C_T^2, \tag{9}$$

which results in

$$\Phi_T = Ak^{-5/3} \approx 0.125C_T^2k^{-5/3}. \tag{10}$$

However, Wyngaard et al. (1971) reported the relationship between Φ_T and C_T^2 as

$$\Phi_T = 0.25C_T^2k^{-5/3}. \tag{11}$$

The proportionality coefficient value of 0.25 was justified by the following comment in the references of op. cit.:

"The constant 0.25 stands for $2(2\pi)^{-1}\Gamma(5/3)\sin(\pi/3)$. Note that we use a range of 0 to $+\infty$ for k_1, whereas a $-\infty$ to $+\infty$ range is used on p. 25 of Ref 1."

where "Ref. 1" is Tatarskii (1961) and k_1 is the analog of our k.

It appears that this value of 0.25 was an error. Since there is no explicit derivation of Eq. (10) presented by Tatarskii (1961), it seems likely that Wyngaard et al. (1971) misinterpreted the limits of integration used to arrive at Tatarskii's formula [our Eq. (10)] and included the extraneous factor of 2. However, as follows from our derivation demonstrated above, the adjustment is already made in Eq. (3) to change the range from 0 to ∞. This means that the expression (11) from Wyngaard et al. (1971) incorporates a proportionality coefficient that is larger than its actual value by a factor of 2.

3. Discussion

The relationship between Φ_T and C_T^2. in the Wyngaard et al. (1971) form, Eq. (11), has been widely used in structure-parameter calculations (e.g., Kaimal 1973; Asimakopoulos et al. 1976; Wyngaard and LeMone 1980; Moulsley et al. 1981; Kohsiek 1982; Cuijpers and Kohsiek 1989; Beland 1993; Green et al. 1994; Muschinski et al. 2001, 2004; Cheinet and Siebesma 2009; Wilson and

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Fedorovich 2012; Maronga et al. 2013; Maronga 2014; Maronga et al. 2014; Gibbs et al. 2016). The error reported here may affect conclusions regarding the validity of this relationship as compared with observational and numerical data. Specifically, the majority of these cited studies found reasonable agreement between this relationship and various direct methods. Thus, the theoretical shortcomings of the method were implicitly corrected through the use of an improper proportionality coefficient.

Beyond the derivation presented above, we have further reason to believe that Eq. (10) is the correct form of the integral spectral approximation of the temperature structure parameter. In Wyngaard [2010, their Eqs. (15.43) and (16.64), p. 374], a specific form of our Eq. (4) is considered that is subsequently integrated to arrive at an equivalent expression for Eq. (8). Although the approximation (10) is not presented in op. cit., one can easily show that the corresponding coefficient of proportionality is also 0.125 instead of the widely adopted value of 0.25 from Wyngaard et al. (1971). We recommend that Eq. (10) be used going forward.

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