Chapter 1

Practicum 1: Flux profile calculations in the atmospheric surface layer based on multi-level measurement data

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1. Turbulence scales in the atmospheric surface layer

- Friction velocity, $u_* = (-\overline{u'w'})^{1/2}$, where u' and w' are, respectively, turbulent fluctuations of the horizontal and vertical velocities and the overbar signifies (Reynolds) averaging over the ensemble of turbulent fluctuations, is employed as **turbulence velocity scale** in the atmospheric surface layer (ASL) under the usual ASL assumption that wind is directed along the shear stress. The vertical variation of kinematic momentum flux $\overline{u'w'}$ (which is negative of shear stress divided by density) is relatively small within the surface layer. Thus, $\overline{u'w'}$ characterizes the whole near-surface portion of the boundary-layer flow and is usually regarded as **surface** kinematic momentum flux.
- Hereafter, the overbars will be omitted in the notation for mean (Reynolds-averaged) velocity, temperature, humidity and associated meteorological variables.
- Near-surface (vertical kinematic) turbulent fluxes of heat and humidity are given by $\overline{w'\theta'}$ (where θ' is the turbulent fluctuation of the potential temperature) and $\overline{w'q'}$ (where q' is the turbulent fluctuation of the specific humidity), respectively. They are used together with the friction velocity to construct the surface-layer **temperature** and **humidity** turbulence **scales**: $\theta_* = -\overline{w'\theta'}/u_*$ and $q_* = -\overline{w'q'}/u_*$. Changes of $\overline{w'\theta'}$ and $\overline{w'q'}$ with height in the idealized (stationary and horizontally homogeneous) ASL flow are relatively small and near-surface values of both fluxes are considered representative of the whole atmospheric surface layer.
- In the ASL flow analyses, it is convenient to introduce also the buoyancy turbulence scale $b_* = -\overline{w'b'}/u_*$, where buoyancy b is defined as $b = -(g/\rho_c)(\rho - \rho_c) = (g/\theta_{vc})(\theta_v - \theta_{vc})$ and subscript *c* denotes reference values of density ρ and virtual potential temperature θ_v.

- Taking into account that $\beta\overline{w'\theta_v'} = \overline{w'b'} = -u_*b_* = -\beta u_*\theta_{v*}$, where $\beta = g/\theta_{vc}$ is the **buoyancy parameter**, b_* can be expressed in terms of the virtual potential temperature scale θ_{v*} as $b_* = \beta\theta_{v*}$. By using

$$-u_*b_* = \beta\overline{w'\theta_v'} = \beta\overline{w'\theta'} + 0.61g\overline{w'q'} = -\beta u_*\theta_* - 0.61gu_*q_* = -u_*(\beta\theta_* + 0.61gq_*),$$

it can further be expressed through the temperature and humidity scales as $b_* = \beta\theta_* + 0.61gq_*$. Since $\beta = g/\theta_{vc} \approx g/\theta_c$, it also follows from the above relationships that $\theta_{v*} = \theta_* + 0.61\theta_c q_*$.

- **Summary of surface-layer scales**: $u_* = (-\overline{u'w'})^{1/2}$ (for velocity), $\theta_{v*} = -\overline{w'\theta_v'}/u_*$ (for virtual potential temperature), $\theta_* = -\overline{w'\theta'}/u_*$ (for potential temperature), $q_* = -\overline{w'q'}/u_*$ (for humidity), and $b_* = -\overline{w'b'}/u_*$ (for buoyancy).
- **Note** that signs of temperature, humidity, and buoyancy scales are opposite to those of fluxes and therefore coincide with signs of the corresponding vertical gradients.

Under **unstable** (convective) conditions:	$\overline{w'\theta_v'}$>0, $\partial\theta_v/\partial z < 0$, and θ_{v*}<0.
	$\overline{w'b'}$>0, $\partial b/\partial z < 0$, and b_*<0.
In the **stable** surface layer:	$\overline{w'\theta_v'}$<0, $\partial\theta_v/\partial z > 0$, and θ_{v*}>0.
	$\overline{w'b'}$<0, $\partial b/\partial z > 0$, and b_*>0.
Under **neutral** conditions:	$\overline{w'\theta_v'}$=0, $\partial\theta_v/\partial z = 0$, and θ_{v*}=0.
	$\overline{w'b'}$=0, $\partial b/\partial z = 0$, and b_*=0.

2. The Monin-Obukhov similarity hypothesis; Monin-Obukhov length

- Fundamental underlying assumption of the Monin-Obukhov hypothesis: at $z >> z_0$ in the atmospheric surface layer, the turbulence regime on all scales of motion except for the dissipation range depends only on distance *z* from the surface and kinematic fluxes of momentum $\overline{u'w'} = -u_*^2$ and buoyancy $\overline{w'b'} = \beta\overline{w'\theta_v'} = -\beta u_*\theta_{v*} = -u_*b_*$.
- The **Monin-Obukhov hypothesis** states that in the surface layer flow at $z >> z_0$ the vertical gradients of (mean) meteorological variables *u*, θ_v θ, *q*, and *b* as well as turbulence statistics of these variables (turbulence moments) are universal functions of dimensionless height *z*/*L* when they are normalized by the corresponding surface-layer turbulence scales (u_*, θ_{v*}, θ_*, q_* and b_*, see above the definitions of these scales) and length scale *L*.
- This length scale *L* is called the **Monin-Obukhov length**. It is introduced (according to fundamental assumption of the Monin-Obukhov theory, see above) as a combination of the surface momentum and buoyancy fluxes:

$$L = -\frac{u_*^3}{\kappa\overline{w'b'}} = -\frac{u_*^3}{\kappa\beta\overline{w'\theta_v'}} = -\frac{(-\overline{u'w'})^{3/2}}{\kappa\beta\overline{w'\theta_v'}}.$$

- The Monin-Obukhov length can also be expressed in terms of surface layer scales as

$$L = \frac{u_*^2}{\kappa b_*} = \frac{u_*^2}{\kappa\beta\theta_{v*}} = \frac{u_*^2}{\kappa(\beta\theta_* + 0.61gq_*)}.$$

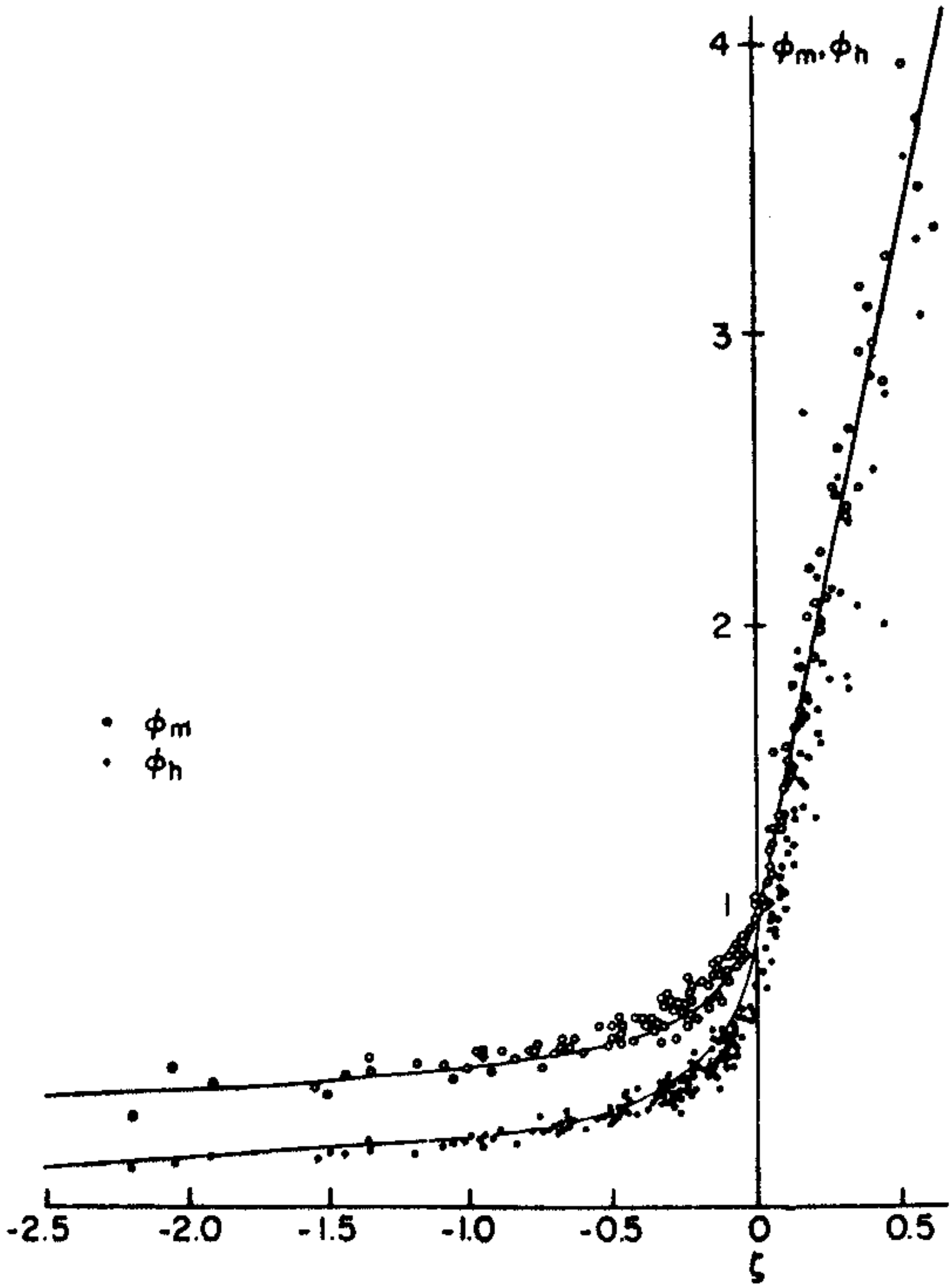
- In the case of dry atmosphere:

$$L = \frac{u_*^2}{\kappa\beta\theta_*}.$$

3. Universality of dimensionless gradients of meteorological variables

- According to the Monin-Obukhov hypothesis,

$$\frac{L}{u_*}\frac{\partial u}{\partial z} = \frac{\partial (u/u_*)}{\partial (z/L)} = \varphi_m{}'(z/L)\,,$$



Universal functions φ_m and φ_h of dimensionless height ζ =*z*/*L*

$$\frac{L}{\theta_*}\frac{\partial \theta}{\partial z} = \frac{\partial (\theta/\theta_*)}{\partial (z/L)} = \varphi_h{}'(z/L)\,,$$

$$\frac{L}{q_*}\frac{\partial q}{\partial z} = \frac{\partial(q/q_*)}{\partial(z/L)} = \varphi_q'(z/L),$$

$$\frac{L}{\theta_{v*}}\frac{\partial \theta_v}{\partial z} = \frac{\partial(\theta_v/\theta_{v*})}{\partial(z/L)} = \varphi_{hv}'(z/L),$$

$$\frac{L}{b_*}\frac{\partial b}{\partial z} = \frac{\partial(b/b_*)}{\partial(z/L)} = \varphi_b'(z/L),$$

where φ_m', φ_h', φ_q', φ_{hv}', and φ_b' are **universal functions** of dimensionless height $\zeta \equiv z/L$.

- The above relationships can be rewritten in the following way:

$$\frac{\kappa z}{u_*}\frac{\partial u}{\partial z} = \kappa\frac{z}{L}\varphi_m'(z/L) \equiv \varphi_m(\zeta),$$

$$\frac{\kappa z}{\theta_*}\frac{\partial \theta}{\partial z} = \kappa\frac{z}{L}\varphi_h'(z/L) \equiv \varphi_h(\zeta),$$

$$\frac{\kappa z}{q_*}\frac{\partial q}{\partial z} = \kappa\frac{z}{L}\varphi_q'(z/L) \equiv \varphi_q(\zeta),$$

$$\frac{\kappa z}{\theta_{v*}}\frac{\partial \theta_v}{\partial z} = \kappa\frac{z}{L}\varphi_{hv}'(z/L) \equiv \varphi_{hv}(\zeta),$$

$$\frac{\kappa z}{b_*}\frac{\partial b}{\partial z} = \kappa\frac{z}{L}\varphi_b'(z/L) \equiv \varphi_b(\zeta),$$

where φ_m, φ_h, φ_q, φ_{hv}, and φ_b are some other universal functions of the dimensionless height $\zeta \equiv z/L$.

- In the neutral surface layer (where $|L| \to \infty$), ζ =*z*/*L*=0 and $\frac{\kappa z}{u_*}\frac{\partial u}{\partial z}$=1, and we have $\varphi_m(0)$ =1. Heat and water vapor in this case are transported as passive scalars and this transport should be independent of *L*. Therefore, corresponding universal functions φ_h, φ_q, φ_{hv}, and φ_b should become constants. This yields logarithmic profiles of temperature, buoyancy, and humidity under (quasi-)neutral conditions in the ASL. For instance, $\frac{\kappa z}{u_*}\frac{\partial u}{\partial z}$=1 integrates to $u = \frac{u_*}{\kappa}\ln z + C$.
- Measurements of the vertical gradients of *u*, θ, and *q* in the ASL generally support predictions of the Monin-Obukhov similarity theory. Experimental data suggest that $\varphi_h \approx \varphi_q$. Examples of measured φ_m and φ_h functions are shown in the plot from Sorbjan (1989) reproduced above.

4. Empirical approximations of Monin-Obukhov universal functions

- A series of specialized surface-layer experiments have been conducted in the 1960s and 1970s in different countries to prove/refute the Monin-Obukhov theory (or to determine limits of its applicability) and to obtain analytical approximations for $\varphi_m(\zeta)$ and $\varphi_h(\zeta)$.
- Numerous sets of analytical approximations for the Monin-Obukhov universal functions have been proposed. Two most commonly used sets are those of Businger *et al*. (1971) and Dyer (Dyer and Hicks 1970, Dyer 1974), see corresponding references in Sorbjan (1989).

Convective (unstable) surface layer ($\zeta = z/L \le 0$).

Businger *et al.*: $\varphi_m(z/L)=\left(1-15\frac{z}{L}\right)^{-1/4}$, $\varphi_h(z/L)=0.74\left(1-9\frac{z}{L}\right)^{-1/2}$, κ =0.35.

Dyer: $\varphi_m(z/L)=\left(1-16\frac{z}{L}\right)^{-1/4}$, $\varphi_h(z/L)=\left(1-16\frac{z}{L}\right)^{-1/2}$, κ =0.4 (originally, 0.41).

Stable surface layer ($\zeta = z/L \geq 0$).

Businger *et al.*: $\varphi_m(z/L)=1+4.7\frac{z}{L}$, $\varphi_h(z/L)=0.74+4.7\frac{z}{L}$, κ =0.35.

Dyer: $\varphi_m(z/L)=1+5\frac{z}{L}$, $\varphi_h(z/L)=1+5\frac{z}{L}$, κ =0.4 (originally, 0.41).

Note that Dyer's set provides C_h =1, while Businger's set provides C_h =0.74.

5. Turbulent exchange coefficients in terms of universal functions

- In the ASL flow, kinematic fluxes of momentum and heat are related to gradients of the corresponding mean fields through the turbulent exchange coefficients as $k(\partial u/\partial z)=-\overline{u'w'}=u_*^2$, where k is the turbulent exchange coefficient for momentum (it is often called *eddy viscosity*) and $k_h(\partial\theta/\partial z)=-\overline{w'\theta'}=u_*\theta_*$, where k_h is the turbulent exchange coefficient for momentum (it is often called *eddy diffusivity*).
- Combining $k(\partial u/\partial z)=-\overline{u'w'}=u_*^2$ and $\frac{\kappa z}{u_*}\frac{\partial u}{\partial z}=\varphi_m(\zeta)$, we have:

$$k(z)=\frac{\kappa u_* z}{\varphi_m(\zeta)}=\kappa u_* L\frac{\zeta}{\varphi_m(\zeta)},$$

which for the neutral conditions (ζ =*z*/*L*=0) provides $k(z)=\kappa u_* z$.

- Using Dyer's expressions of φ_m for unstable conditions and stable conditions (see above), we have

$k(z)=\kappa u_* z(1-16z/L)^{1/4}$ for **unstable conditions**, ζ =*z*/*L*≤0, and

$k(z)=\frac{\kappa u_* z}{1+5z/L}$ for **stable conditions**, ζ =*z*/*L*≥0).

- Taking into account that $k_h(\partial\theta/\partial z)=-\overline{w'\theta'}=u_*\theta_*$ and $\frac{\kappa z}{\theta_*}\frac{\partial\theta}{\partial z}=\varphi_h(\zeta)$, see sections **2** and **3**, we obtain the following expression for the turbulent heat exchange coefficient

$$k_h(z)=\frac{\kappa u_* z}{\varphi_h(\zeta)}=\kappa u_* L\frac{\zeta}{\varphi_h(\zeta)}.$$

- **Note** that because $\varphi_q(\zeta) \approx \varphi_h(\zeta)$ the turbulent exchange coefficient for humidity $k_q(z)$ is approximately equal to $k_h(z)$.
- In terms of Dyer's universal functions:

$k_h(z)=\kappa u_* z(1-16z/L)^{1/2}$ for **unstable conditions**, ζ =*z*/*L*≤0, and

$k_h(z)=\frac{\kappa u_* z}{1+5z/L}$ for **stable conditions**, ζ =*z*/*L*≥0).

- **Note** that under stable conditions, the considered approximations of the universal functions provide equality of the exchange coefficients for momentum and heat $k_h(z)=k(z)$. Under neutral conditions, when ζ =*z*/*L*=0: $k_h(z)=k(z)=\kappa u_* z$.

- Based on the above relationships, the turbulent Prandtl number $\mathrm{Pr}_t = k / k_h$ can be expressed in terms of universal functions $\varphi_m(\zeta)$ and $\varphi_h(\zeta)$ as $\mathrm{Pr}_t(\zeta) = \varphi_h / \varphi_m$. With Dyer's functions, this provides $\mathrm{Pr}_t(\zeta) = (1-16z/L)^{-1/4}$ in the unstable surface layer (ζ =*z*/*L*≤0), and $\mathrm{Pr}_t(\zeta)$ =1 in the stable surface layer (ζ =*z*/*L*≥0).
- Due to $\varphi_q(\zeta) \approx \varphi_h(\zeta)$, the turbulent Schmidt number $\mathrm{Sc}_t(\zeta) = k / k_q$ is approximately equal to the turbulent Prandtl number $\mathrm{Pr}_t(\zeta)$.
- **Note** that under neutral conditions: $\mathrm{Pr}_t(0) = \mathrm{Sc}_t(0) = \varphi_h(0) / \varphi_m(0) = C_h$.

6. Relationships between *z*/*L* and Richardson numbers

- Richardson numbers, specified as

$$\mathrm{Ri}_f = \frac{\beta \overline{w'\theta_v'}}{\overline{u'w'}(\partial u/\partial z)} = \frac{\overline{w'b'}}{\overline{u'w'}(\partial u/\partial z)}$$ (**flux Richardson number**) and

$$\mathrm{Ri} = \frac{\beta(\partial \theta_v / \partial z)}{(\partial u/\partial z)^2} = \frac{\partial b/\partial z}{(\partial u/\partial z)^2}$$ (**gradient Richardson number**),

where $\mathrm{Ri}_f = \dfrac{k_h}{k}\mathrm{Ri} = \dfrac{\mathrm{Ri}}{\mathrm{Pr}_t}$, characterize proportion between buoyancy and shear contributions to the turbulence kinetic energy production in a turbulent flow.

- The following sequence of relationships is worth of memorizing: $\mathrm{Pr}_t \approx \mathrm{Sc}_t = \varphi_h / \varphi_m = k / k_h = \mathrm{Ri}/\mathrm{Ri}_f$.
- In terms of Dyer's functions, under unstable conditions, when ζ =*z*/*L*≤0: Ri=$\dfrac{z}{L} = \zeta$ ≤0 (because $\varphi_h = \varphi_m{}^2$) and $\mathrm{Ri}_f = \zeta(1-16\zeta)^{1/4}$≤0; under stable conditions, when (ζ =*z*/*L*≥0): Ri= $\mathrm{Ri}_f = \zeta/(1+5\zeta)$ ≥0.
- **Note** that in the latter case ζ =$\mathrm{Ri}/(1-5\mathrm{Ri})$ at Ri=0.2 corresponds to the infinitely large positive ζ (or infinitesimal positive *L*) that is the case of extreme stability when turbulence cannot exist. In other words, Dyer's approximation yields the critical Richardson number value Ri_c =0.2.

Exercise 1

1. Based on $\varphi_h = \varphi_q$, show that $\varphi_{hv} = \varphi_b = \varphi_h$.
2. Obtain expressions Ri=$\dfrac{\varphi_h}{\varphi_m{}^2}\dfrac{z}{L} = \dfrac{\mathrm{Pr}_t}{\varphi_m}\zeta$ and $\mathrm{Ri}_f = \dfrac{1}{\varphi_m}\dfrac{z}{L} = \dfrac{\mathrm{Pr}_t}{\varphi_h}\zeta$ taking into account that $\varphi_b \approx \varphi_h$ and $\mathrm{Pr}_t = \varphi_h / \varphi_m$.
3. Based on Dyer's universal functions, obtain the following expressions for k and $k_h = k_q$ as functions of Ri:

$k(z)$= $\kappa u_* z(1-16\mathrm{Ri})^{1/4}$, $k_h(z)$= $\kappa u_* z(1-16\mathrm{Ri})^{1/2}$ for ζ =*z*/*L*≤0, Ri≤0,

$k(z)$= $k_h(z)$= $\kappa u_* z(1-5\mathrm{Ri})$ for ζ =*z*/*L*≥0, Ri≥0.

4. Expanding Dyer's $\varphi_m(\zeta)$ and $\varphi_h(\zeta)$ for $\zeta \le 0$ in the Maclaurin series around $\zeta = 0$ and neglecting terms of the order higher than 1, obtain the following approximations of $\varphi_m(\zeta)$ and $\varphi_h(\zeta)$ for $\zeta \le 0$ and $|\zeta| << 1$: $\varphi_m(\zeta) = 1+4\zeta$ and $\varphi_h(\zeta)$ =$1+8\zeta$. Find values of $\zeta < 0$, at

which differences between the above linear approximations $\varphi_m(\zeta)$ and $\varphi_h(\zeta)$ and regular Dyer's universal functions exceed 10%.

7. Integral forms of flux-profile relationships

- The dimensionless gradients of velocity, temperature, and humidity, which are universal functions of $\zeta \equiv z/L$, can be integrated over z to obtain the explicit expressions of the corresponding profiles.
- Integration of $\varphi_m(z/L) = \dfrac{\kappa z}{u_*}\dfrac{\partial u}{\partial z}$ between levels z_1 and $z > z_1$ in the surface layer leads to the following expression for the wind velocity profile: $u(z) = u(z_1) + \dfrac{u_*}{\kappa}\left[\ln\dfrac{z}{z_1} - \psi_m\left(\dfrac{z}{L}, \dfrac{z_1}{L}\right)\right]$,

  where $\psi_m\left(\dfrac{z}{L}, \dfrac{z_1}{L}\right) = \psi_m(\zeta, \zeta_1) = \int_{z_1}^{z}[1-\varphi_m(z/L)]d\ln z = \int_{\zeta_1}^{\zeta}[1-\varphi_m(\zeta)]d\ln\zeta$.
- If the lower integration level is taken to be the surface roughness height (length) z_0, where the mean flow velocity is assumed to be zero, the wind profile appears as $u(z) = \dfrac{u_*}{\kappa}\left[\ln\dfrac{z}{z_0} - \psi_m\left(\dfrac{z}{L}, \dfrac{z_0}{L}\right)\right]$. The latter expression indicates that function $\psi_m\left(\dfrac{z}{L}, \dfrac{z_0}{L}\right) = \psi_m(\zeta, \zeta_0)$ describes the deviation of the velocity profile from the logarithmic law due to the effect of atmospheric stability/instability. It is commonly called the **stability correction function,** or simply **stability correction**.
- In practical applications, $\zeta_0 = z_0/L$ in $\psi_m(\zeta, \zeta_0)$ is often replaced by zero and the stability correction is taken as $\Psi_m(\zeta) \equiv \psi_m(\zeta, 0)$, so that the velocity profile has the following approximate form:

$$u(z) = \frac{u_*}{\kappa}\left[\ln\frac{z}{z_0} - \Psi_m\left(\frac{z}{L}\right)\right].$$

- Dyer's universal functions $\varphi_m(\zeta)$ provide (see **Exercise 2**)

$\Psi_m(\zeta) = 2\ln\dfrac{1+x}{2} + \ln\dfrac{1+x^2}{2} - 2\tan^{-1}x + \dfrac{\pi}{2}$, where $x = (1-16\zeta)^{1/4}$, for $\zeta \le 0$ (unstable flow) and

$\Psi_m(\zeta) = -5\zeta$ for $\varphi_m(\zeta) = 1+5\zeta$ for $\zeta \ge 0$ (stable flow).

- Integration of the universal function $\varphi_h(\zeta)$ between levels z_1 and $z > z_1$ leads to

$$\theta(z) = \theta(z_1) + \frac{\theta_*}{\kappa}\left[\ln\frac{z}{z_1} - \psi_h\left(\frac{z}{L}, \frac{z_1}{L}\right)\right] \text{ and}$$

$$q(z) = q(z_1) + \frac{q_*}{\kappa}\left[\ln\frac{z}{z_1} - \psi_h\left(\frac{z}{L}, \frac{z_1}{L}\right)\right],$$

where $\psi_h\left(\dfrac{z}{L}, \dfrac{z_1}{L}\right) = \psi_h(\zeta, \zeta_1) = \int_{z_1}^{z}[1-\varphi_h(z/L)]d\ln z = \int_{\zeta_1}^{\zeta}[1-\varphi_h(\zeta)]d\ln\zeta$.

- Using the concepts of roughness lengths for temperature and specific humidity ($\theta = \theta_s$ at $z = z_{0\theta}$, $q = q_s$ at $z = z_{0q}$), we can express the temperature and humidity profiles as

$$\theta(z) = \theta_s + \frac{\theta_*}{\kappa}\left[\ln\frac{z}{z_{0\theta}} - \psi_h\left(\frac{z}{L}, \frac{z_{0\theta}}{L}\right)\right] \text{ and } q(z) = q_s + \frac{q_*}{\kappa}\left[\ln\frac{z}{z_{0q}} - \psi_h\left(\frac{z}{L}, \frac{z_{0q}}{L}\right)\right].$$

- Approximate forms of these profiles are

$$\theta(z) = \theta_s + \frac{\theta_*}{\kappa}\left[\ln\frac{z}{z_{0\theta}} - \Psi_h\left(\frac{z}{L}\right)\right] \text{ and}$$

$$q(z) = q_s + \frac{q_*}{\kappa}\left[\ln\frac{z}{z_{0q}} - \Psi_h\left(\frac{z}{L}\right)\right],$$

where $\Psi_h(\zeta) \equiv \psi_h(\zeta, 0)$.

- If $\varphi_h(\zeta)$ is taken after Dyer (see section **4**), the corresponding integral function is $\Psi_h(\zeta) = 2\ln\frac{1+y}{2}$, where $y = (1-16\zeta)^{1/2}$, for $\zeta \le 0$ (unstable conditions) and $\Psi_h(\zeta) = -5\zeta$ for $\zeta \ge 0$ (stable conditions), see **Exercise 2**.

Exercise 2

1. Show that Dyer's universal functions $\varphi_m(\zeta)$ provide

$\Psi_m(\zeta) = 2\ln\frac{1+x}{2} + \ln\frac{1+x^2}{2} - 2\tan^{-1}x + \frac{\pi}{2}$, where $x = (1-16\zeta)^{1/4}$, for $\zeta \le 0$ (unstable flow) and

$\Psi_m(\zeta) = -5\zeta$ for $\varphi_m(\zeta) = 1+5\zeta$ for $\zeta \ge 0$ (stable flow).

2. Show that $\varphi_h(\zeta)$ after Dyer provides $\Psi_h(\zeta) = 2\ln\frac{1+y}{2}$, where $y = (1-16\zeta)^{1/2}$, for $\zeta \le 0$ (unstable conditions) and $\Psi_h(\zeta) = -5\zeta$ for $\zeta \ge 0$ (stable conditions).

8. Calculation of surface fluxes from meteorological measurements at two levels

- In sections **3** and **4** we obtained in following expressions, which relate the surface layer turbulence scales u_*, θ_*, and q_* (and therefore, surface layer vertical kinematic turbulent fluxes of momentum: $\overline{w'u'} = -u_*^2$, heat: $\overline{w'\theta'} = -u_*\theta_*$, and humidity: $\overline{w'q'} = -u_*q_*$) to gradients of corresponding meteorological variables: $\frac{\kappa z}{u_*}\frac{\partial u}{\partial z} = \varphi_m(\zeta)$, $\frac{\kappa z}{\theta_*}\frac{\partial \theta}{\partial z} = \frac{\kappa z}{q_*}\frac{\partial q}{\partial z} = \varphi_h(\zeta)$, where φ_m and φ_h are universal functions of dimensionless height ζ=*z*/*L*. After Dyer, these functions may be approximated as $\varphi_m(\zeta) = (1-16\zeta)^{-1/4}$, $\varphi_h(\zeta) = (1-16\zeta)^{-1/2}$ for $\zeta \le 0$ and $\varphi_m(\zeta) = \varphi_h(\zeta) = 1+5\zeta$ for $\zeta \ge 0$.
- Now imagine that we have mean (Reynolds-averaged) values of *u*, *T* (absolute temperature), and *q* measured at two heights in the surface layer: z_1 and z_2, with $z_2 > z_1$. This gives us three pairs of quantities: (u_1, u_2), ($\theta_1 \approx T_1$, $\theta_2 \approx T_2$), and (q_1, q_2), where subscripts denote corresponding measurement levels. We can also calculate finite differences of these variables across the layer $\Delta z = z_2 - z_1$: $\Delta u = u_2 - u_1$, $\Delta\theta = \theta_2 - \theta_1$, and $\Delta q = q_2 - q_1$.

- We have to define a level between z_1 and z_2, to which values of the finite gradients and $\mathrm{Ri} = \beta \frac{\Delta\theta_v \Delta z}{\Delta u^2}$, where $\beta = \frac{g}{\theta_{vc}}$ is the buoyancy parameter, can be referred to in this case. Based on the fact that gradients of meteorological variables in the surface layer decrease fast with distance from the surface (in the neutral case they decrease as 1/*z*), the reference level for Ri is usually specified as $z_s = \sqrt{z_1 z_2}$. It is also possible to take $z_s = (z_2 - z_1)/\ln(z_2/z_1)$, which is the height where $\Delta u/\Delta z = \partial u/\partial z$ in the case of perfectly logarithmic profile (please demonstrate it yourself).
- The reference value of virtual potential temperature θ_{vc} in $\beta = \frac{g}{\theta_{vc}}$ may be taken constant, for instance, θ_{vc} =300 K.
- For calculation of actual (dynamic) turbulent fluxes (which are expressed through their kinematic counterparts as $\rho \overline{w'u'}$, $\rho c_p \overline{w'\theta'}$, and $\rho \overline{w'q'}$) we will also need the values of air density ρ and specific heat at constant pressure c_p =1004 J kg^{-1} K^{-1}. Due to small vertical variations of air density in the surface layer, ρ can be evaluated from *p* (usually known) and *T* at one of measurement levels. For instance $\rho = p/(RT_1)$, if we take temperature at the first measurement level.

Flux calculation algorithm

1. The Richardson number at the reference level z_s is evaluated from the approximate relationship:

$$\mathrm{Ri}(z_s) = \frac{\beta(\Delta\theta/\Delta z) + 0.61g(\Delta q/\Delta z)}{(\Delta u/\Delta z)^2}, \text{ where } z_s = \sqrt{z_1 z_2}.$$

2. If Ri(z_s)≥0.2, further derivations make no sense because the value of Ri is beyond the critical limit.
3. If Ri(z_s)<0.2, we proceed with calculation of dimensionless height $\zeta_s = z_s/L$ that is related to Richardson number Ri(z_s) as

   ζ_s =Ri(z_s) if Ri(z_s)≤0 (unstable stratification) and

   ζ_s =Ri(z_s)/[1−5Ri(z_s)] if Ri(z_s)≥0 (stable stratification), see section **6**.
4. From ζ_s, the value of Monin-Obukhov length scale *L* can be calculated as $L = z_s/\zeta_s$. In the present algorithm, *L* is a supplementary parameter.
5. The calculated ζ_s enters the expressions of the universal functions φ_m and φ_h:

$\varphi_m(\zeta_s) = (1-16\zeta_s)^{-1/4}$ if Ri(z_s)≤0 (unstable), $\varphi_m(\zeta_s) = 1+5\zeta_s$ if Ri(z_s)≥0 (stable);

$\varphi_h(\zeta_s) = (1-16\zeta_s)^{-1/2}$ if Ri(z_s)≤0 (unstable), $\varphi_h(\zeta_s) = 1+5\zeta_s$, if Ri($z_s$)≥0 (stable).

6. From the universal function, we calculate the surface layer turbulence velocity, temperature, and humidity scales from $u_* = \frac{\kappa z_s}{\varphi_m(\zeta_s)} \frac{\Delta u}{\Delta z}$, $\theta_* = \frac{\kappa z_s}{\varphi_h(\zeta_s)} \frac{\Delta \theta}{\Delta z}$, and $q_* = \frac{\kappa z_s}{\varphi_h(\zeta_s)} \frac{\Delta q}{\Delta z}$, where κ =0.4 is the von Kármán constant.

7. The kinematic surface turbulent fluxes are calculated from u_*, θ_*, and q_* as $\overline{w'u'} = -u_*^2$ (momentum), $\overline{w'\theta'} = -u_*\theta_*$ (heat), and $\overline{w'q'} = -u_*q_*$ (humidity).

8. Finally, we obtain the surface vertical turbulent fluxes of momentum, $\rho\overline{w'u'}$, heat, $\rho c_p \overline{w'\theta'}$, and humidity, $\rho\overline{w'q'}$.

Exercise 3

You are given four datasets with mean velocity, temperature, and specific humidity values measured at two levels in the atmospheric surface layer.

Set 1. Measurement levels: z_1=0.5m and z_2=2m. Data: u_1=3m/s, u_2=4m/s, T_1=36ºC, T_2=29ºC, q_1=0.008, q_2=0.003, p_1=1000hPa.

Set 2. Measurement levels: z_1=2m and z_2=8m. Data: u_1=4m/s, u_2=8m/s, T_1=20ºC, T_2=22ºC, q_1=0.004, q_2=0.006, p_1=1000hPa.

Set 3. Measurement levels: z_1=1m and z_2=4m. Data: u_1=3m/s, u_2=6m/s, T_1=15ºC, T_2=15ºC, q_1=0.009, q_2=0.009, p_1=1000hPa.

Set 4. Measurement levels: z_1=4m and z_2=9m. Data: u_1=2m/s, u_2=3m/s, $T_1 = -2$ ºC, T_2=8ºC, q_1=0.001, q_2=0.005, p_1=1000hPa.

For each of the above datasets (as long as physical limitations allow):

a. Determine class of stability (unstable, stable, or neutral), and evaluate corresponding value of *L*;

b. Calculate the surface layer turbulence scales and turbulent fluxes of momentum, heat, humidity, and buoyancy;

c. Find values of turbulent exchange coefficients for momentum, k, and heat, k_h, and calculate turbulent Prandtl number at $z_s = \sqrt{z_1 z_2}$;

d. Calculate mean wind velocity, temperature, and specific humidity at z_s and *z*=10m.

9. Calculation of surface turbulent fluxes in the case of non-coinciding measurement levels

- In this case, we have mean values of *u*, *T* (absolute temperature), and *q* measured at following levels: u_1, u_2 at z_{u1}, z_{u2} ($z_{u2} > z_{u1}$), $T_1 \sqcup \theta_1$, $T_2 \sqcup \theta_2$ at $z_{\theta 1}$, $z_{\theta 2}$ ($z_{\theta 2} > z_{\theta 1}$), and q_1, q_2 at z_{q1}, z_{q2} ($z_{q2} > z_{q1}$).
- Like in the previously considered case of two-level measurements (see section **8**), c_p =1004J·kg^{-1}·K^{-1}, atmospheric pressure is assumed to be known, and the buoyancy parameter is $\beta = g/\theta_{vc}$ with θ_{vc}=300 K. **Note** that, like in the previous case, this is only one of several possible ways of evaluating θ_{vc} in this case. The air density can be calculated as $\rho = p/(RT_1)$.

Flux calculation algorithm

1. In a first approximation, the profiles of *u*, θ, and *q* in the surface layer may be taken logarithmic. Thus, we may express the increments of variables as $u_2 - u_1 = \dfrac{u_*}{\kappa}\ln\dfrac{z_{u2}}{z_{u1}}$, $\theta_2 - \theta_1 = \dfrac{\theta_*}{\kappa}\ln\dfrac{z_{\theta 2}}{z_{\theta 1}}$, and $q_2 - q_1 = \dfrac{q_*}{\kappa}\ln\dfrac{z_{q2}}{z_{q1}}$. These expressions provide first approximations for the surface layer turbulence scales u_*, θ_*, and q_*.

2. Based of the calculated turbulence scales, the Monin-Obukhov length is evaluated as $L = \dfrac{u_*^{\,2}}{\kappa(\beta\theta_* + 0.61 g q_*)}$.

3. If $z_e / |L|$ <<1, where z_e is the highest measurement level of the three (z_{u2}, $z_{\theta 2}$, z_{q2}), the stratification of the surface layer may be considered neutral. One make take, for instance, $z_e / |L|$ =0.01 as the lowest limit for the non-neutral case. After that, the kinematic fluxes can be directly evaluated from scales u_*, θ_*, and q_* as $\overline{w'u'} = -u_*^{\,2}$ (momentum), $\overline{w'\theta'} = -u_*\theta_*$ (heat), and $\overline{w'q'} = -u_* q_*$ (humidity).

4. If $z_e / |L|$ ≥0.01, we have to calculate new approximations of u_*, θ_*, and q_* from

$$u_2 - u_1 = \frac{u_*}{\kappa}\left[\ln\frac{z_{u2}}{z_{u1}} - \Psi_m\left(\frac{z_{u2}}{L}\right) + \Psi_m\left(\frac{z_{u1}}{L}\right)\right],$$

$$\theta_2 - \theta_1 = \frac{\theta_*}{\kappa}\left[\ln\frac{z_{\theta 2}}{z_{\theta 1}} - \Psi_h\left(\frac{z_{\theta 2}}{L}\right) + \Psi_h\left(\frac{z_{\theta 1}}{L}\right)\right],$$

$$q_2 - q_1 = \frac{q_*}{\kappa}\left[\ln\frac{z_{q2}}{z_{q1}} - \Psi_h\left(\frac{z_{q2}}{L}\right) + \Psi_h\left(\frac{z_{q1}}{L}\right)\right],$$

taking into account the sign of *L* and using appropriate integral functions from section **7**.

5. With new scales u_*, θ_*, and q_* we calculate new approximation for $L = \dfrac{u_*^{\,2}}{\kappa(\beta\theta_* + 0.61 g q_*)}$.

6. Steps 4 and 5 are repeated until the relative difference between new and old values of *L* becomes reasonably small (let say, of the order of 0.01)

7. Based on the resulting values of u_*, θ_*, and q_*, the turbulent fluxes are calculated using $\overline{w'u'} = -u_*^{\,2}$, $\overline{w'\theta'} = -u_*\theta_*$, and $\overline{w'q'} = -u_* q_*$, and then multiplying kinematic fluxes by $c_p = 1004 \mathrm{J} \cdot \mathrm{kg}^{-1} \cdot \mathrm{K}^{-1}$ and ρ.

8. Finally, velocity, temperature, and humidity at any level *z* within the surface layer can be obtained from

$$u(z) = u_1 + \frac{u_*}{\kappa}\left[\ln\frac{z}{z_{u1}} - \Psi_m\left(\frac{z}{L}\right) + \Psi_m\left(\frac{z_{u1}}{L}\right)\right],$$

$$\theta(z) = \theta_1 + \frac{\theta_*}{\kappa}\left[\ln\frac{z}{z_{\theta 1}} - \Psi_h\left(\frac{z}{L}\right) + \Psi_h\left(\frac{z_{\theta 1}}{L}\right)\right],$$

$$q(z) = q_1 + \frac{q_*}{\kappa}\left[\ln\frac{z}{z_{q1}} - \Psi_h\left(\frac{z}{L}\right) + \Psi_h\left(\frac{z_{q1}}{L}\right)\right].$$

Note that for such evaluation one can use velocity, temperature, and humidity values from any measurement level (for instance, u_2, θ_2, and q_2 along with corresponding measurement levels may be used instead of u_1, θ_1, and q_1).

10. Retrieval of surface roughness length values from the gradient measurements

- In the case, when the lower measurement levels in the surface layer are taken as (or assumed to be) roughness heights (lengths) $z_{u1} = z_0$, $z_{\theta 1} = z_{0\theta}$, $z_{q1} = z_{0q}$, at which, according to the definitions of roughness lengths, the meteorological variables reach their surface values *u*=0, $\theta = \theta_s$, and $q = q_s$, the flux-profile relationships can be written as

$$u(z_{u2}) = \frac{u_*}{\kappa}\left[\ln\frac{z_{u2}}{z_0} - \Psi_m(\zeta_{u2}) + \Psi_m(\zeta_0)\right],$$

$$\theta(z_{\theta 2}) = \theta_s + \frac{\theta_*}{\kappa}\left[\ln\frac{z_{\theta 2}}{z_{0\theta}} - \Psi_h(\zeta_{\theta 2}) + \Psi_h(\zeta_{0\theta})\right],$$

$$q(z_{q2}) = q_s + \frac{q_*}{\kappa}\left[\ln\frac{z_{q2}}{z_{0q}} - \Psi_h(\zeta_{q2}) + \Psi_h(\zeta_{0q})\right].$$

- These expressions can be used for calculation of surface-layer turbulence scales and turbulent fluxes from meteorological measurements at a single level in the surface layer. However, for such calculation we need values of z_0, $z_{0\theta}$, z_{0q}, θ_s, and q_s, which generally are not very easy to obtain.
- On the other hand, given the surface values of temperature θ_s and humidity q_s, as well as velocity, temperature, and humidity turbulence scales (determined, for instance, from the two-level measurements in the surface layer), the above expressions can be used for evaluation of surface roughness lengths z_0, $z_{0\theta}$, and z_{0q}.

Exercise 4

You are given two sets of meteorological variables measured at different levels in the atmospheric surface layer.

Set 1. Measurement levels: $z_{u1} = z_{\theta 1} = z_{q1}$=0.5m and $z_{u2} = z_{\theta 2} = z_{q2}$=2m. Data: u_1=3m/s, u_2=4m/s, T_1=36ºC, T_2=29ºC, q_1=0.008, q_2=0.003, $p_{\theta 1}$=1000hPa. For this dataset:

a. Calculate the surface turbulent fluxes employing the algorithm described in section **9**.

b. Estimate mean velocity, temperature, specific humidity, turbulent exchange coefficients, and Ri at z=10m.

c. Compare results with your calculations for the Set 1 in **Exercise 3**.

Set 2. Measurement levels: z_{u1}=1m, $z_{\theta 1} = z_{q1}$=2m, z_{u2}=8m, $z_{\theta 2} = z_{q2}$=6m. Data: u_1=2m/s, u_2=8m/s, T_1=8ºC, T_2=11ºC, q_1=0.004, q_2=0.006, $p_{\theta 1}$=1000hPa. For this dataset:

a. Calculate the surface turbulent fluxes employing the algorithm described in **9.**

b. Estimate mean velocity, temperature, specific humidity, turbulent exchange coefficients, and Ri at z=10m.

References

Garratt, J. R., 1994: *The Atmospheric Boundary Layer*, Cambridge University Press, 316pp.
Sorbjan, Z., 1989: *Structure of the Atmospheric Boundary Layer*, Prentice Hall, 317 pp.
Stull, R. B., 1988: *An Introduction to Boundary Layer Meteorology*. Kluwer, 666 pp.