Rayleigh Scattering

• Fundamental in understanding particle scattering

• Widely used in radar meteorology

\[ \eta = \int \sigma_s(D)N(D)dD = \frac{\pi^5}{\lambda^4} \frac{m^2 - 1}{m^2 + 2} \ Z_e \]

• Continuously be useful

• Valid regime

Historical development

Ishimaru, 2-5

• The law of Rayleigh (1871) scattering: When light is scattered by particles which are very small compared with any of the wavelengths, the ratio of the amplitudes of the vibrations of the scattered and incident light varies inversely as the square of the wavelength and the intensity of the lights themselves as the inverse forth power

• Derived by matching dimension

\[ \frac{A_s}{A_i} \propto V \cdot r \cdot \lambda \cdot c \cdot \rho \propto \frac{V}{\lambda^2 r} \]

\[ \frac{S_s}{S_i} \propto \frac{V^2}{\lambda^3} \]

• Allow explanation of blue sky and red sun

• Unresolved issues: absorption, angle and polarization dependence
Understood as a dipole radiation

- Scattering

\[ \vec{E}_s(r) = \nabla \times \nabla \times \vec{\Pi}_s(r) \]

\[ \vec{\Pi}_s = \frac{1}{\varepsilon_0} \int_V G(\vec{r}, \vec{r}') \vec{P}(\vec{r}') d\vec{r}' \]

- Dipole oscillator

\[ \hat{e}_s f(\hat{k}_s, \hat{k}_i) = \frac{k^2}{4\pi\varepsilon_0} \int_V \left\{ -\hat{k}_s \times [\hat{k}_s \times \vec{P}(\vec{r})] e^{-ik_s \cdot \vec{r}'} \right\} d\vec{r}' \]
Dielectric sphere excited by incident wave field

- Internal field
  \[ \vec{E} = \frac{3}{\varepsilon_r + 2} \vec{E}_i \]
- Polarization
  \[ \vec{P} = \varepsilon_0 (\varepsilon_r - 1) \vec{E} = 3 \frac{\varepsilon_0 (\varepsilon_r - 1)}{\varepsilon_r + 2} \vec{E}_i \]
- Scattering amplitude
  \[ \hat{e}_s f(\hat{k}_s,\hat{k}_i) = \frac{k^2}{4\pi} \frac{3(\varepsilon_r - 1)}{\varepsilon_r + 2} V[-\hat{k}_s \times (\hat{k}_s \times \vec{e}_i)] \]
  \[ = k^2 a^3 \frac{\varepsilon_r - 1}{\varepsilon_r + 2} [-\hat{k}_s \times (\hat{k}_s \times \vec{e}_i)] \]

Scattering pattern

- Differential cross section
  \[ \sigma_d(\hat{k}_s,\hat{k}_i) = \left( k^2 a^3 \right) \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \sin^2 \chi \]
- Scattering field pattern
  \[ \sin \chi \]
- Scattering power pattern
  \[ \sin^2 \chi \]

Parallel to scattering plane  Perpendicular to scattering plane
Plot with “polar”

Scattering, radar, absorption cross sections

- Scattering cross section
  \[
  \sigma_s = \int_{4\pi} \sigma_a(\hat{k}_i, \hat{k}_j) d\Omega
  \]
  \[
  = k^4 a^6 \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) \int_0^\pi \int_0^{2\pi} \sin^2 \chi \sin \chi d\chi d\phi = \frac{8\pi k^4 a^6}{3} \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right)^2
  \]

- Radar cross section
  \[
  \sigma_b = 4\pi k^4 a^6 \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right)^2
  \]

- Absorption cross section
  \[
  \sigma_a = \frac{P_s}{S_j} = k \varepsilon'' \left( \frac{3}{\varepsilon_r + 2} \right)^2
  \]
  \[
  V = \frac{4}{3} \pi a^3 k \varepsilon'' \left( \frac{3}{\varepsilon_r + 2} \right)^2
  \]
  \[
  \sigma_t = \sigma_a + \sigma_s
  \]
  Not \( \sigma_i = \frac{4\pi}{k} \text{Im} \left( f(\hat{k}_i, \hat{k}_j) \right) \) if \( \varepsilon'' = 0 \)

- Scattering albedo
  \[
  w_0 = \frac{\sigma_s}{\sigma_t}
  \]
Valid regime

- Scattering $D < \lambda/15$
- Absorption/extinction $D < \lambda/50$