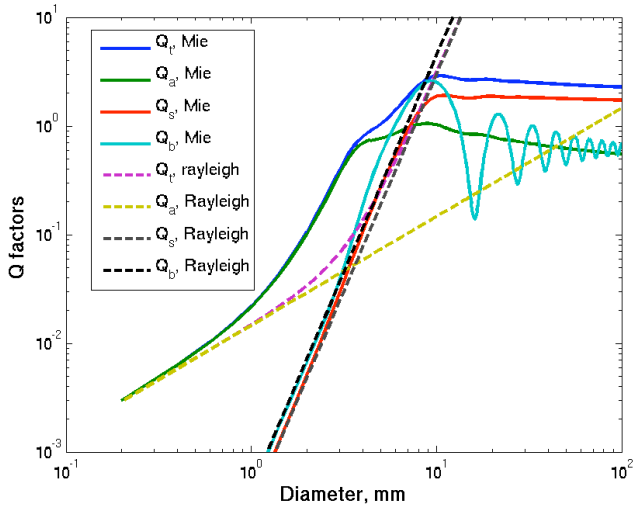
Valid regimes for Rayleigh, Mie and geometric optics



- Rayleigh: Scattering $D < \lambda/15$, Absorption/extinction $D < \lambda/50$
- Geometric optics: $D > 100\lambda$

Mie Theory (1908)

Ishimaru, 2-8;
Bohern and Huffman (1983)

- Exact solution of plane wave scattering by a sphere
- Accurate calculation of radar reflectivity and bistatic scattering pattern
- Determine valid regime for Rayleigth scattering approximation.

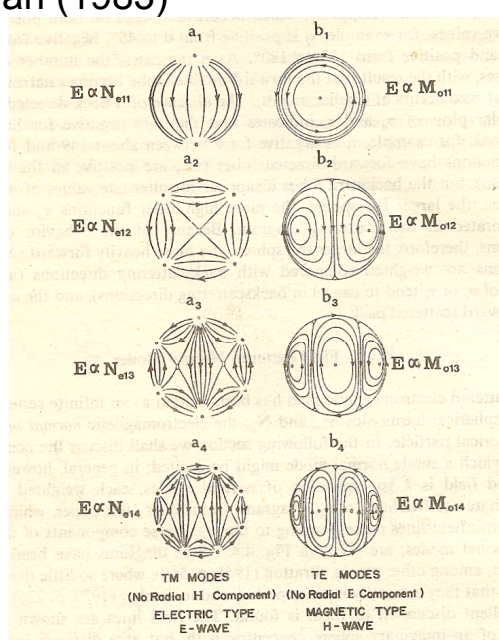
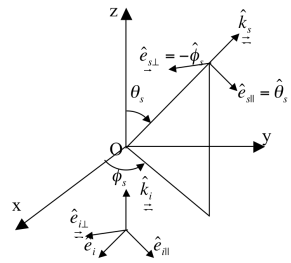


Figure 4.4 Electric field patterns: normal modes (Mie, 1908).

Boundary problem



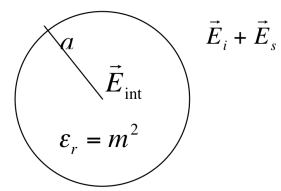
- Incident fields

$$\vec{E}_i = e^{ikz}\hat{x} \qquad \vec{H}_i = \frac{1}{\eta}e^{ikz}\hat{y}$$

- Boundary problem

$$\hat{r}\times\vec{E}_{\text{int}}\big|_{r=a} = \hat{r}\times[\vec{E}_i + \vec{E}_s]\big|_{r=a}$$

$$\hat{r}\times\vec{H}_{\text{int}}\big|_{r=a} = \hat{r}\times[\vec{H}_i + \vec{H}_s]\big|_{r=a}$$



- Component form

$$E_{\text{int}\theta}\big|_{r=a} = [E_{i\theta} + E_{s\theta}]\big|_{r=a} \qquad E_{\text{int}\phi}\big|_{r=a} = [E_{i\phi} + E_{s\phi}]\big|_{r=a}$$

$$H_{\text{int}\theta}\big|_{r=a} = [H_{i\theta} + H_{s\theta}]\big|_{r=a} \qquad H_{\text{int}\phi}\big|_{r=a} = [H_{i\phi} + H_{s\phi}]\big|_{r=a}$$

Two approaches to represent wave fields

- In terms of vector spherical harmonics Bohern and Huffman (1983), Ch. 4

  Born and Wolf (2001), Ch. 14
- By radial component of Hertz vectors for incident, scattered and internal wave fields

$$\vec{E} = \nabla\times\nabla\times(\Pi_1\vec{r}) + i\omega\mu_0\nabla\times(\Pi_2\vec{r})$$

$$\vec{H} = -i\omega\varepsilon\nabla\times(\Pi_1\vec{r}) + \nabla\times\nabla\times(\Pi_2\vec{r})$$

Solution of wave equation

- Wave equation

$$\nabla^2\Pi + k^2\Pi = 0$$

- In spherical coordinate

$$\frac{1}{r}\frac{\partial^2(r\Pi)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Pi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Pi}{\partial\phi^2} + k^2\Pi = 0$$

- Assuming a function form

$$\Pi = R(r)\Theta(\theta)\Phi(\phi)$$

$$\Pi_{mn}(kr,\theta,\phi) = \psi_n(kr)P_n^m(\cos\theta)e^{im\phi}$$

Expressions of wave fields

- Expansion

$$r\Pi_{1s} = \frac{-1}{k^2}\sum_{n=1}^{\infty}\frac{i^{n-1}(2n+1)}{n(n+1)}a_n\zeta_n(kr)P_n^1(\cos\theta)\cos\phi$$

$$r\Pi_{2s} = \frac{-1}{\eta k^2}\sum_{n=1}^{\infty}\frac{i^{n-1}(2n+1)}{n(n+1)}b_n\zeta_n(kr)P_n^1(\cos\theta)\sin\phi$$

- Solve for a_n, b_n by matching the boundary condition

$$a_n = \frac{\psi_n(ka)\psi_n'(kma) - m\psi_n(kma)\psi_n'(ka)}{\zeta_n(ka)\psi_n'(kma) - m\psi_n(kma)\zeta_n'(ka)}$$

$$b_n = \frac{m\psi_n(ka)\psi_n'(kma) - \psi_n(kma)\psi_n'(ka)}{m\zeta_n(ka)\psi_n'(kma) - \psi_n(kma)\zeta_n'(ka)}$$

Scattered wave fields

- Expression

$$\begin{bmatrix} E_{s\perp} \\ E_{s\parallel} \end{bmatrix} = \frac{e^{ikr}}{-ikr} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} E_{i\perp} \\ E_{i\parallel} \end{bmatrix}$$

$$\vec{E}_s = \frac{e^{ikr}}{-ikr} \left[S_2(\theta)\cos\phi\hat{\theta} - S_1(\theta)\sin\phi\hat{\phi} \right]$$

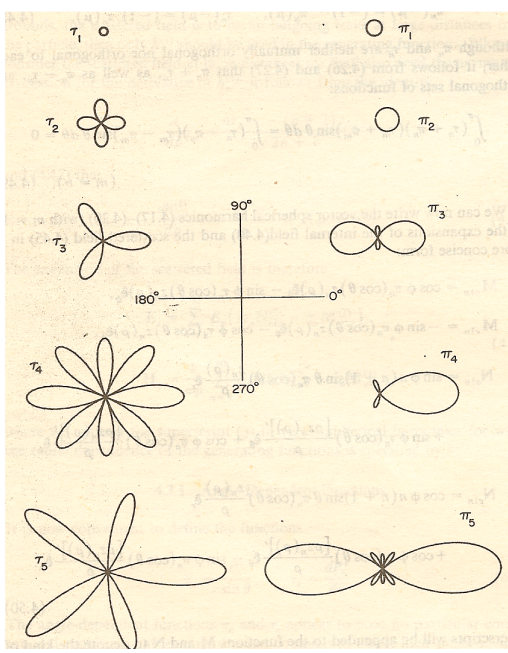
- Angle dependence

$$S_1(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \pi_n(\cos\theta) + b_n \tau_n(\cos\theta) \right]$$

$$S_2(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \tau_n(\cos\theta) + b_n \pi_n(\cos\theta) \right]$$

$$\pi_n(\cos\theta) = \frac{P_n^1(\cos\theta)}{\sin\theta} \qquad \tau_n(\cos\theta) = \frac{d}{d\theta} P_n^1(\cos\theta)$$

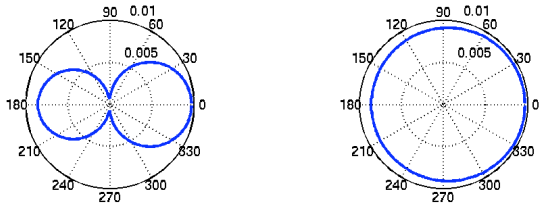
Pattern functions



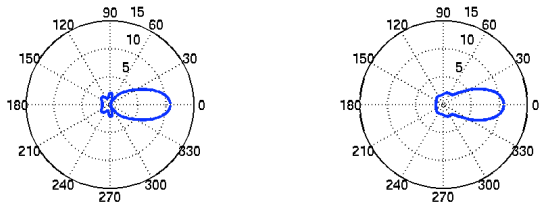
From Bohren & Huffman, 1983

X-band scattering pattern

differ from Rayleigh: |sinχ|



D = 2 mm: |S(θ)|



D = 2 cm: |S(θ)|

Cross sections

- Extinction cross section

$$\sigma_t = \frac{4\pi}{k}\mathrm{Im}\left(\frac{S(0)}{-ik}\right) = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\mathrm{Re}[a_n + b_n]$$

- Scattering cross section

$$\sigma_s = \frac{1}{k^2}\int_{4\pi}\left(|S_1(\theta)|^2\sin^2\phi + |S_2(\theta)|^2\cos^2\phi\right)d\Omega = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\left(|a_n|^2 + |b_n|^2\right)$$

- Radar cross section

$$\sigma_b = \frac{\pi}{k^2}\left|\sum_{n=1}^{\infty}(2n+1)(-1)^n\left(a_n - b_n\right)\right|^2$$