Valid regimes for Rayleigh, Mie and geometric optics

- Rayleigh: Scattering $D < \lambda/15$, Absorption/extinction $D < \lambda/50$
- Geometric optics: $D > 100\lambda$

Mie Theory (1908)
Ishimaru, 2-8; Bohern and Huffman (1983)

- Exact solution of plane wave scattering by a sphere
- Accurate calculation of radar reflectivity and bistatic scattering pattern
- Determine valid regime for Rayleigh scattering approximation.
Boundary problem

- Incident fields
  \[ \vec{E}_i = e^{ikz} \hat{x}, \quad \vec{H}_i = \frac{1}{\eta} e^{ikz} \hat{y} \]

- Boundary problem
  \[ \hat{r} \times \vec{E}_{\text{int}}|_{r=a} = \hat{r} \times [\hat{E}_i + \vec{E}_s]|_{r=a} \]
  \[ \hat{r} \times \vec{H}_{\text{int}}|_{r=a} = \hat{r} \times [\hat{H}_i + \vec{H}_s]|_{r=a} \]

- Component form
  \[ E_{\text{int} \theta}|_{r=a} = [E_{i \theta} + E_{s \theta}]|_{r=a} \quad E_{\text{int} \phi}|_{r=a} = [E_{i \phi} + E_{s \phi}]|_{r=a} \]
  \[ H_{\text{int} \theta}|_{r=a} = [H_{i \theta} + H_{s \theta}]|_{r=a} \quad H_{\text{int} \phi}|_{r=a} = [H_{i \phi} + H_{s \phi}]|_{r=a} \]

Two approaches to represent wave fields

- In terms of vector spherical harmonics Bohern and Huffman (1983), Ch. 4
  Born and Wolf (2001), Ch. 14
- By radial component of Hertz vectors for incident, scattered and internal wave fields
  \[ \vec{E} = \nabla \times \nabla \times (\Pi_1 \vec{r}) + i \omega \mu_0 \nabla \times (\Pi_2 \vec{r}) \]
  \[ \vec{H} = -i \omega \varepsilon \nabla \times (\Pi_1 \vec{r}) + \nabla \times \nabla \times (\Pi_2 \vec{r}) \]
Solution of wave equation

• Wave equation
  \[ \nabla^2 \Pi + k^2 \Pi = 0 \]

• In spherical coordinate
  \[ \frac{1}{r} \frac{\partial^2 (r \Pi)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Pi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Pi}{\partial \phi^2} + k^2 \Pi = 0 \]

• Assuming a function form
  \[ \Pi = R(r) \Theta(\theta) \Phi(\phi) \]
  \[ \Pi_{mn}(kr, \theta, \phi) = \psi_n(kr) P^m_n(\cos \theta) e^{im\phi} \]

Expressions of wave fields

• Expansion
  \[ r \Pi_1 = \frac{-1}{k^2} \sum_{n=1}^{\infty} i^{n-1} (2n+1) \frac{a_n \xi_n(kr) P_n^1(\cos \theta) \cos \phi}{n(n+1)} \]
  \[ r \Pi_2 = \frac{-1}{\eta k^2} \sum_{n=1}^{\infty} i^{n-1} (2n+1) \frac{b_n \xi_n(kr) P_n^1(\cos \theta) \sin \phi}{n(n+1)} \]

• Solve for \( a_n \), \( b_n \) by matching the boundary condition
  \[ a_n = \frac{\psi_n(ka) \psi'_n(kma) - m \psi_n(kma) \psi'_n(ka)}{\xi_n(ka) \psi'_n(kma) - m \psi_n(kma) \xi'_n(ka)} \]
  \[ b_n = \frac{m \psi_n(ka) \psi'_n(kma) - \psi_n(kma) \psi'_n(ka)}{m \xi_n(ka) \psi'_n(kma) - \psi_n(kma) \xi'_n(ka)} \]
Scattered wave fields

• Expression

\[
\begin{bmatrix}
E_{s\perp} \\
E_{\text{dil}}
\end{bmatrix} = \begin{bmatrix}
e^{ikr} & 0 \\
-i kr & S_2
\end{bmatrix}
\begin{bmatrix}
E_{i\perp} \\
E_{\text{dil}}
\end{bmatrix}
\]

\[
\vec{E}_s = \frac{e^{ikr}}{-i kr} \left[ S_2(\theta) \cos \phi \hat{\theta} - S_1(\theta) \sin \phi \hat{\phi} \right]
\]

• Angle dependence

\[
S_1(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[ a_n \pi_n(\cos \theta) + b_n \tau_n(\cos \theta) \right]
\]

\[
S_2(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[ a_n \pi_n(\cos \theta) + b_n \tau_n(\cos \theta) \right]
\]

\[
\pi_n(\cos \theta) = \frac{P_n'(\cos \theta)}{\sin \theta} \quad \tau_n(\cos \theta) = \frac{d}{d\theta} P_n'(\cos \theta)
\]

Pattern functions

From Bohren & Huffman, 1983
X-band scattering pattern
differ from Rayleigh: $|\sin \chi|$

![Scattering patterns](image)

**Cross sections**

- **Extinction cross section**
  \[
  \sigma_i = \frac{4\pi}{k} \operatorname{Im} \left( \frac{S(0)}{-ik} \right) = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1) \operatorname{Re} [a_n + b_n]
  \]

- **Scattering cross section**
  \[
  \sigma_s = \frac{1}{k^2} \int_{4\pi} \left( |S_i(\theta)|^2 \sin^2 \phi + |S_z(\theta)|^2 \cos^2 \phi \right) d\Omega = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1) (|a_n|^2 + |b_n|^2)
  \]

- **Radar cross section**
  \[
  \sigma_r = \frac{\pi}{k^2} \sum_{n=1}^{\infty} (2n+1)(-1)^n (a_n - b_n)^2
  \]