Optimization and data assimilation

- Linear problem without error
  \[ y = Ax \]
  
  \( A: m \times n \) matrix, \( m > n \): measurements > unknown
  
  rank of matrix: \( p \), \( p < n \): under-determined problem

- Linear problem with error
  
  \[ y = Ax + \delta \]
  
  \[ y_1 = x_1 + 0.99x_2 + \delta_1 \]
  
  \[ y_2 = 0.99x_1 + x_2 + \delta_2 \]

  \( x_1 = \frac{y_1 - \delta_1 - 0.99(y_2 - \delta_2)}{1 - 0.99^2} \)

Linear estimator

- Linear forward model
  
  \[ y = F(x) + \delta = Ax + \delta \]

  \[ p(y \mid x) = \frac{1}{(2\pi)^{n/2} \left| S_\delta \right|^{1/2}} \exp\left[ -\frac{1}{2} (y - Ax)^T S_\delta^{-1} (y - Ax) \right] \]

  \[ -2 \ln p(y \mid x) = (y - Ax)^T S_\delta^{-1} (y - Ax) + c_1 \]

  \[ -2 \ln p(x) = (x - x_a)^T S_a^{-1} (x - x_a) + c_2 \]

  \[ -2 \ln p(x \mid y) = (y - Ax)^T S_\delta^{-1} (y - Ax) + (x - x_a)^T S_a^{-1} (x - x_a) + c_3 \]

- Estimator
  
  \[ \hat{x} = x_a + (A S_\delta^{-1} A^T + S_a^{-1})^{-1} A^T S_\delta^{-1} (y - Ax_a) \]
Nonlinear inversion

• Nonlinear forward problem

\[ y = F(x) \]

• Not exactly a problem of solving a set of nonlinear eqs.
  – Prior information
  – Error characterization
  – Cost function, not quadratic

Degree of nonlinearity

• Linearization

\[ \hat{y} - y_a = F(\hat{x}) - F(x_a) = A(\hat{x} - x_a) + \Delta_y \]

\[ \Delta_y = F(\hat{x}) - F(x_a) - A(\hat{x} - x_a) \]

• Nonlinearity

\[ c^2 = \Delta_y^T S_0^{-1} \Delta_y \]
Newton iteration method

- Extension from linear solution
  \[ \hat{x} = x_a + (AS^{-1}_0 A^T + S^{-1}_a)^{-1} A^T S^{-1}_0 (y - Ax_a) \]
  \[ y - Ax_a \Rightarrow y - F(\hat{x}) - A(\hat{x} - x_a) \]
  \[ \hat{x}_{i+1} = x_a + (AS^{-1}_0 A^T + S^{-1}_a)^{-1} A^T S^{-1}_0 (y - F(\hat{x}_i) - A(\hat{x}_i - x_a)) \]

- Simplified form (Levenberg-Marquardt method)
  \[ \hat{x}_{i+1} = \hat{x}_i + (AA^T + \gamma I)^{-1} A^T (y - F(\hat{x}_i)) \]

Data assimilation as inversion

- Numerical weather prediction: Initialization with observations
  - Not straightforward
  - Observation error
  - Compatibility problem

- As inverse problem, taking both prior information and measurement error into account in retrieval process
Assimilation methods

\[ J = [x - x_a]^T S_a^{-1} [x - x_a] + [y - F(x)]^T S_\delta^{-1} [y - F(x)] \]

- Successive correction

\[ \hat{x}_i - x_{ai} = \sum_k \frac{\sigma^2_w w_{ik}}{\sigma^2_a w_{ik} + \sigma^2_k} (y_k - y_{ak}) \]

- Variational analysis: adjoint method

- Ensemble Kalman filtering

Adjoint method

- Iterative solution

\[ x_{i+1} = x_i - [\nabla_x (\nabla_x J)^\top] \nabla_x J \]
\[ x_{i+1} = x_i - \gamma \nabla_x J \]

- Adjoint solution

\[ x_t = M_t x_{t-1} \]
\[ y_t = F(x_t) + \delta_t = F(M_t ... M_1 x_0) + \delta_t \]
\[ J = \sum_{i=0}^T [(y - F(x_i))^T S_\delta^{-1} [y - F(x_i)]] \]
\[ \nabla_{x_0} J = -\sum_{i=0}^T [\partial F_i(x_i)/\partial x_0]^T S_{\delta,2}^{-1} [y - F(x_i)] = 0 \]