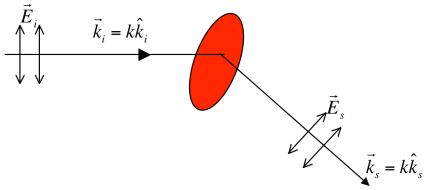
Scattering amplitude and cross section
Ishimaru (1997), 2-1, 2-3



- Three components: scatterer (particle), incident wave, and scattered wave
- Wave fields: vector and complex
- A scalar and real parameter such as cross section not enough to full characterize the scattering properties

Scatterer's properties

$$\varepsilon(\vec{r}) = \varepsilon_r(\vec{r})\varepsilon_0 = \left(\varepsilon'(\vec{r}) + i\varepsilon''(\vec{r})\right)\varepsilon_0$$
$$\mu(\vec{r}) = \mu_0$$

- Physical properties: size, shape, orientation, *composition (density) and temperature.*

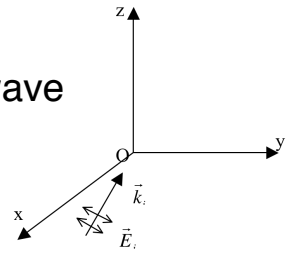
- EM properties

$$\varepsilon(\vec{r}) = \varepsilon_r(\vec{r})\varepsilon_0 = \left(\varepsilon'(\vec{r}) + i\varepsilon''(\vec{r})\right)\varepsilon_0$$
$$\mu(\vec{r}) = \mu_0$$

Plane wave and spherical wave

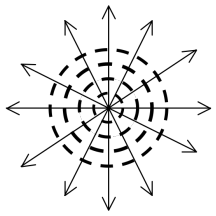
- Linearly polarized plane wave

$$\vec{E} = \hat{e}\ E_0 e^{i\vec{k}\bullet\vec{r}}$$



- Spherical wave in far field ($r > D^2/\lambda$)

$$\vec{E} = \hat{e}\ E_0 \frac{e^{i\vec{k}\bullet\vec{r}}}{r} = \hat{e}\ E_0 \frac{e^{ikr}}{r}$$



Scattering amplitude

- Incident plane wave

$$\vec{E}_i = \hat{e}_i E_0 e^{i\vec{k}_i\bullet\vec{r}}$$

- Amplitude of scattered wave

$$\vec{E}_s = \hat{e}_s f(\hat{k}_s,\hat{k}_i) E_0 \frac{e^{ikr}}{r} = \vec{f}(\hat{k}_s,\hat{k}_i) E_0 \frac{e^{ikr}}{r}$$

Hinted argument?

- Scattered wave amplitude corresponding a unit amplitude of incident wave

Poynting vector: power flux density

- Incident power flux density $\vec{H} = \frac{1}{\eta}\hat{k} \times \vec{E}$

$$\vec{S}_i = \frac{1}{2}\mathrm{Re}\left(\vec{E}_i \times \vec{H}_i^*\right) = \frac{|E_0|^2}{2\eta}\hat{k}_i$$

- Scattering power flux density

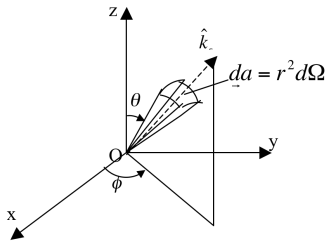
$$\vec{S}_s = \frac{1}{2}\mathrm{Re}\left(\vec{E}_s \times \vec{H}_s^*\right) = \frac{|E_s|^2}{2\eta}\hat{k}_s = \frac{|f(\hat{k}_s,\hat{k}_i)|^2}{r^2}\frac{|E_0|^2}{2\eta}\hat{k}_s$$

Differential power

- Differential power

$$dP_s = S_s da = \frac{|f(\hat{k}_s,\hat{k}_i)|^2}{r^2}\frac{|E_0|^2}{2\eta}r^2 d\Omega$$

$$S_i = \frac{|E_0|^2}{2\eta}$$



- Normalized by incident power flux density

$$\frac{dP_s}{S_i} = |f(\hat{k}_s,\hat{k}_i)|^2 \, d\Omega = \sigma(\hat{k}_s,\hat{k}_i)d\Omega$$

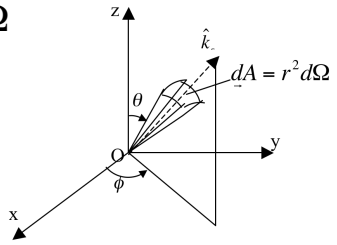
- Differential scattering section

$$\sigma_d(\hat{k}_s,\hat{k}_i) = |f(\hat{k}_s,\hat{k}_i)|^2$$

Scattering cross section

- Total scattered power

$$\frac{P_s}{S_i} = \int |f(\hat{k}_s, \hat{k}_i)|^2 \, d\Omega = \int \sigma_d(\hat{k}_s, \hat{k}_i) d\Omega$$



- Scattering cross section

$$\sigma_s = \int \sigma_d(\hat{k}_s, \hat{k}_i) d\Omega$$

- Radar cross sections

$$\sigma_{bi}(\hat{k}_s, \hat{k}_i) = 4\pi\sigma_d(\hat{k}_s, \hat{k}_i) \qquad \sigma_b = 4\pi\sigma_d(-\hat{k}_i, \hat{k}_i)$$

Absorption cross section

- Power loss inside the particle

$$P_a = \int \frac{1}{2} \omega \varepsilon'' \varepsilon_0 |E_{\text{int}}(\vec{r}\,')|^2 \, d\vec{r}\,'$$

- Absorption cross section

$$\sigma_a = \frac{\int \frac{1}{2} \omega \varepsilon_0 \varepsilon'' \left| \vec{E}(\vec{r}\,') \right|^2 d\vec{r}\,'}{S_i}$$

Total cross section

- Total power loss and total cross section

$$P = P_s + P_a$$

$$\sigma_t = \sigma_s + \sigma_a$$

- Optical theorem

$$\sigma_t = \frac{4\pi}{k}\mathrm{Im}\left(f(\hat{k}_i, \hat{k}_i)\right)$$