C vectors and P vectors

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1. The C-vector form of the quasigeostrophic ω equation

The forcing function in the Q-vector form of the quasigeostrophic ω equation is expressed in terms of the quasihorizontal divergence of a two-dimensional vector, the Q vector. Qin Xu recently showed how the quasigeostrophic ω equation can be expressed in terms of the vertical component of the curl of a three-dimensional vector, the C vector. He also showed how the Q vector is related to the C vector. The C-vector form of the ω equation is a three-dimensional extension of the Q-vector ω equation.

We define

$$C_1 \equiv Q_2 \tag{1}$$

$$C_2 \equiv -Q_1 \tag{2}$$

From (1) and (2) it can be can be seen that

$$\mathbf{C_H} = \mathbf{Q} \times \mathbf{k} = C_1 \mathbf{i} + C_2 \mathbf{j} \tag{3}$$

Thus, the horizontal component of the C vector ($\mathbf{C_H}$*) is perpendicular and to the right of the Q vector, and of the same magnitude as the Q vector.*

Consider the frictionless form of the quasigeostrophic equations of motion (5.7.43) and (5.7.44) expressed as follows:

$$(\partial/\partial t + u_g\partial/\partial x + v_g\partial/\partial y)\, u_g = f_o\, v_a + \beta\, y\, v_g \tag{4}$$

$$(\partial/\partial t + u_g\partial/\partial x + v_g\partial/\partial y)\, v_g = -f_o\, u_a - \beta\, y\, u_g \tag{5}$$

Instead of forming a vorticity equation as we did when we derived the Q-vector formulation of the quasigeostrophic ω equation, we now form a divergence equation. Differentiating (4) with respect to x, and adding the result to the derivative of (5) with respect to y, we find that

$$\begin{aligned}\partial/\partial x(f_o^2\, v_a) - \partial/\partial y\, (f_o^2\, u_a) &= - f_o^{\,2}\, (\partial u_g/\partial x\ \partial v_g/\partial y - \partial v_g/\partial x\ \partial u_g/\partial y)\\ &\quad + \beta\, f_o\, u_g - f_o\, \beta\, y\, \zeta g\end{aligned} \tag{6}$$

which, with the neglect of the β terms, we define as 2 C_3. The former two mathematical operations are equivalent to forming a divergence equation; however, since the divergence of the "theoretician's" geostrophic wind is zero (5.7.49), the time derivative term involving the geostrophic wind vanishes. It follows from (6) that

$$C_3 = f_o^{\,2}\, \zeta_a / 2 \ \propto\ \zeta_a \tag{7}$$

The vertical component of the C vector (C3) is proportional to the vorticity of the ageostrophic part of the wind. Equation (7) can be expanded, by using the definition of C3 and (6), in terms of geostrophic vorticity (5.6.1) and the geostrophic resultant deformation ((3.1.66), (5.7.36) and (5.7.37)) as follows:

$$C3 = f_o / 4 \, (D_g^2 - \zeta_g^2) = (f_o^2 / 2) \, \zeta_a \qquad (8)$$

The vertical component of the C vector is therefore also related to the geostrophic resultant deformation and the geostrophic vorticity. Equation (8) is a useful way to compute the ageostrophic vorticity from the geostrophic wind field. However, it provides no information about the more important *divergent* part of the ageostrophic wind.

Consider the equations (5.7.50) and (5.7.51), which represent one step in the derivation of the Q-vector formulation of the quasigeostrophic w equation, expressed as follows in terms of the components of the Q vector (5.7.55) (recall that it has been assumed that s is independent of x and y):

$$\partial\omega/\partial y - f_o^2/\sigma \, \partial v_a/\partial p = - 2Q_2 - (R/\sigma p) \, \beta \, y \, \partial T/\partial x \qquad (9)$$

$$\partial\omega/\partial x - f_o^2/\sigma \, \partial u_a/\partial p = - 2Q_1 + (R/\sigma p) \, \beta \, y \, \partial T/\partial y \qquad (10)$$

We scale the vertical height coordinate by $(N/f_o)^2$ (see the handout on the vertical scaling of the quasigeostrophic ω and height-tendency equations) and denote the three dimensional wind vector involving the ageostrophic wind and vertical velocity by $\mathbf{v}_{a\omega} = \mathbf{v}_a - (\omega/\rho g)\mathbf{k}$. Using (9) and (10) without the β terms, and using the definition of the horizontal components of the C vector, (1) and (2), we see that the components of the C vector can be expressed in terms of the ageostrophic vertical circulation $\mathbf{v}_a$ - ω as follows:

$$f_o^2/\sigma \, \partial v_a/\partial p - \partial\omega/\partial y = 2 \, C_1 \quad \propto \quad \mathbf{i} \bullet \nabla \times \mathbf{v}_{a\omega} \qquad (11)$$

$$- f_o^2/\sigma \, \partial u_a/\partial p + \partial\omega/\partial x = 2 \, C_2 \quad \propto \quad \mathbf{j} \bullet \nabla \times \mathbf{v}_{a\omega} \qquad (12)$$

$$\partial v_a/\partial x - \partial u_a/\partial y = 2 \, C_3 / f_o^2 = \mathbf{k} \bullet \nabla \times \mathbf{v}_{a\omega} \qquad (13)$$

The C vector may therefore be interpreted as the three-dimensional vorticity vector associated with the ageostrophic, vertical circulation (Fig. 1).

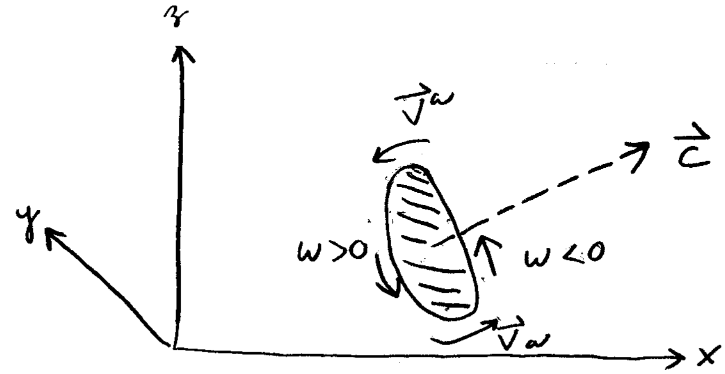

Fig. 1. The circulations associated with the horizontal and vertical vorticity of the ageostrophic vertical circulation.

The equation of continuity (5.7.53) can be written as

$$\partial u_a/\partial x + \partial v_a/\partial y + \partial\omega/\partial p = 0 \tag{14}$$

Differentiating (12) with respect to x, and subtracting from it the result of (11) differentiated with respect to y, and retaining the β terms in (9) and (10), and using (14), we obtain the following form of the quasigeostrophic ω equation:

$$(\nabla_p^2 + f_o^2/\sigma\ \partial^2/\partial p^2)\,\omega = 2\,\mathbf{k} \bullet \nabla\ \mathbf{X}\ \mathbf{C}\ - R/\sigma p\ \beta\ \partial T/\partial x \tag{15}$$

[Alternatively, we could have derived this equation easily by using (3), solving for **Q**, and plugging the resultant expression ($\mathbf{Q} = \mathbf{k}\ \mathbf{X}\ \mathbf{C_H}$) into the Q-vector form of the quasigeostrophic ω equation.] In this form, quasigeostrophic vertical motion is induced by the vertical component of the vorticity of C. We note an interesting parallel between the C-vector and Q-vector formulations of the quasigeostrophic ω equation: The forcing for the former is the vertical vorticity of a vector field, while the forcing for the latter is the divergence of a vector field. In other words, the C-vector formulation involves forcing by the nondivergent part of a vector field, while the Q-vector formulation involves forcing by the irrotational part of a vector field. Since the horizontal component of the C vector and Q vector have the same magnitude, and since the horizontal component of the C vector is orthogonal to the Q vector, the former contains the same information as the Q vector. However, the three-dimensional C vector also contains information about the tilt of the ageostrophic circulation. It has therefore been suggested that it may be more versatile than the Q vector.

Consider, for example, diffluence in the geostrophic wind (geostrophic stretching deformation) acting on a temperature gradient (Fig. 2). We can perform our analysis first by using natural coordinates for Q vectors to compute the Q vectors, and then rotate the Q vectors to the right by 90° to get the C_H vectors; alternatively, we can compute the C_H vectors directly by working in a natural coordinate system for C_H vectors. Q vectors are directed from the warm to the cold side; convergence (divergence) of Q and rising (sinking) motion are found on the cold (warm) side. Let us now analyze the problem using C vectors: The horizontal component of C in this example points in the downstream direction. The vertical component of the curl of $\mathbf{C_H}$ is cyclonic (positive) on the left side of the flow, and anticyclonic (negative) on the right side. Thus, according to (15) there is rising (sinking) motion on the cyclonic (anticyclonic), cold (warm) side. Thus, the C-vector analysis is so far consistent with the Q-vector analysis.

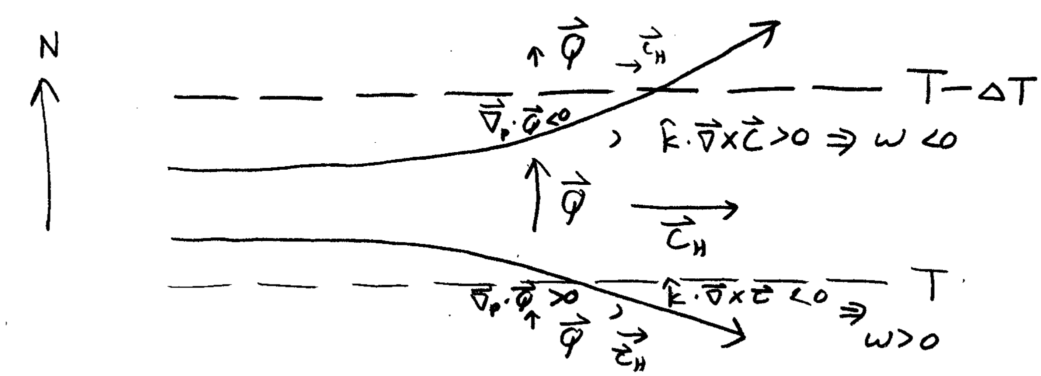

Fig. 2. Illustration of the use of the C-vector field to diagnose vertical velocity and ageostrophic circulation. Geostrophic wind (streamlines), isotherms (dashed lines).

However, we know from (8) that since there is no geostrophic vorticity, and there is nonzero resultant geostrophic deformation, that

$$C_3 \propto D_g^2 > 0$$

i.e., the vertical component of the circulation is cyclonic. Hence the three-dimensional C vector tilts upward to the right, and the ageostrophic circulation is tilted as shown in Fig. 3; in other words, the plane of the vertical circulation leans to the west (towards lower values of x) with height. The reader should, as an exercise, consider the Q-vector field shown in Fig. 5.27c, and (a) show that the C-vector form of the quasigeostrophic ω equation yields the same vertical-motion field as the Q-vector form of the quasigeostrophic ω equation, and (b) that the plane of the vertical circulation leans toward the warm air at low centers (and toward the cold air at high centers).

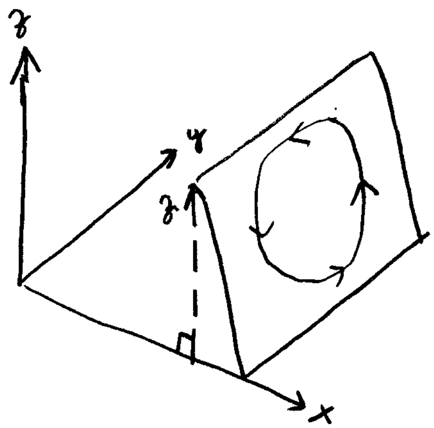

Fig. 3. Illustration of the tilt of the plane of the ageostrophic vertical circulation.

So, we use the geostrophic wind to calculate the resultant geostrophic deformation and geostrophic vorticity. Using (8), we find C_3, from which we can find the tilt of the ageostrophic vertical circulation and the ageostrophic vorticity.

2. The P-vector form of the quasigeostrophic height-tendency equation

The P-vector form of the quasigeostrophic height-tendency equation allows one to diagnose height tendency from a vector field, just as the Q and C-vector forms of the quasigeostrophic ω equation allow one to diagnose vertical motion from vector fields.

We define the three-dimensional P vector, from which we can use the geostrophic wind relations (5.5.7) and (5.5.8), and the equation of continuity (5.7.49), to show that its horizontal components can be expressed as follows:

$$P_1 \equiv \partial v_g/\partial x \bullet \nabla\Phi = - f_o \mathbf{v_g} \bullet \boldsymbol{\nabla}_p v_g \qquad (16)$$

$$P_2 \equiv \partial v_g/\partial y \bullet \boldsymbol{\nabla}\Phi = f_o \mathbf{v_g} \bullet \boldsymbol{\nabla}_p u_g \qquad (17)$$

We use the same geostrophic wind relations and the thermal-wind relations (4.1.120) and (4.1.121), to show that its vertical component can be expressed as follows:

$$P_3 \equiv f_o^2/\sigma \ \partial \mathbf{v}_g/\partial p \bullet \boldsymbol{\nabla}\Phi = f_o^2/\sigma \ R/p \ \mathbf{v}_g \bullet \boldsymbol{\nabla}_p T \qquad (18)$$

Thus, it is seen that the horizontal components of **P** represent geostrophic advection of geostrophic momentum, and the vertical component of **P** represents geostrophic advection of temperature (or thickness).

Consider the frictionless form of the quasigeostrophic equations of motion (4) and (5) expressed, using the “theoretician’s definition” of the geostrophic wind (5.5.7) and (5.5.8), as follows:

$$\partial/\partial t\,(- 1/ f_o\ \partial\Phi/\partial y) + \mathbf{v}_g \bullet \nabla u_g = f_o v_a + \beta y v_g \qquad (19)$$

$$\partial/\partial t\,(1/ f_o\ \partial\Phi/\partial x) + \mathbf{v}_g \bullet \nabla v_g = - f_o u_a - \beta y u_g \qquad (20)$$

The adiabatic form of the quasigeostrophic thermodynamic equation (5.5.19), expressed using the hydrostatic equation (5.6.2), is

$$\partial/\partial t\,(-p/R\ \partial\Phi/\partial p) + \mathbf{v}_g \bullet \boldsymbol{\nabla}_p T = \omega\, \sigma\, p/R \qquad (21)$$

Solving for u_a, v_a, and ω from (19), (20), and (21), respectively, and using the definition of geopotential-height tendency (5.6.4), we find using the definition of the P vector (16) - (18) that

$$u_a = P_1 / f_o^2 - 1 / f_o^2 \, \partial\chi/\partial x - \beta y u_g / f_o \qquad (22)$$

$$v_a = P_2 / f_o^2 - 1 / f_o^2 \, \partial\chi/\partial y - \beta y v_g / f_o \qquad (23)$$

$$\omega = P_3 / f_o^2 - 1/\sigma \, \partial\chi / \partial p \qquad (24)$$

Let us assume that the static-stability parameter σ is independent of pressure. Differentiating (22), (23), and (24) with respect to x, y, and p respectively, and adding the expressions up, we obtain the following P-vector form of the quasigeostrophic height-tendency equation:

$$(\nabla_p^2 + f_o^2 / \sigma \; \partial^2/\partial p^2)\, \chi = \boldsymbol{\nabla}_p \bullet \mathbf{P} + \partial P_3 / \partial p - f_o \beta v_g \qquad (25)$$

The height tendency can therefore be related to the three-dimensional divergence of the P-vector field: Divergence (convergence) of **P** is therefore associated with falling (rising) heights.

Differentiating (22) and (23) separately with respect to x and y, respectively, and adding the resulting expressions together, we find that

$$\partial\zeta_g / \partial t = 1 / f_o \; \boldsymbol{\nabla}_p \bullet \mathbf{P} - f_o \, \delta - \beta v_g \qquad (26)$$

Comparing (26) to the quasigeostrophic vorticity equation (5.5.20), we find that

$$- \mathbf{v}_g \bullet \boldsymbol{\nabla} \zeta_g = 1 / f_o \, \boldsymbol{\nabla}_p \bullet \mathbf{P} \qquad (27)$$

Thus, the quasihorizontal divergence of the P-vector field is equivalent to geostrophic advection of geostrophic vorticity. It is easily seen using (18) that the vertical derivative of the vertical component of P is equivalent to differential (mean) temperature advection, i. e.,

$$\partial P_3 / \partial p = - f_o^2 / \sigma \;\; \partial/\partial p \; [- (R / p) \, \mathbf{v}_g \bullet \; \mathbf{\nabla}_p \, T] \tag{28}$$

Therefore in an equivalent barotropic or barotropic atmosphere, $P_3 = 0$, and in the absence of the b term we only have to consider the forcing function $\mathbf{\nabla}_p \bullet \mathbf{P}$. From the definitions of the horizontal components of the P vector (16) and (17) in terms of the geopotential-height gradient ($\mathbf{\nabla}\Phi$), we find that they can be expressed more easily in a natural coordinate system for the horizontal components of **P**, i.e., one in which the x axis is oriented along the geopotential-height contours, and the y axis is directed toward lower heights as below in Fig. 4.

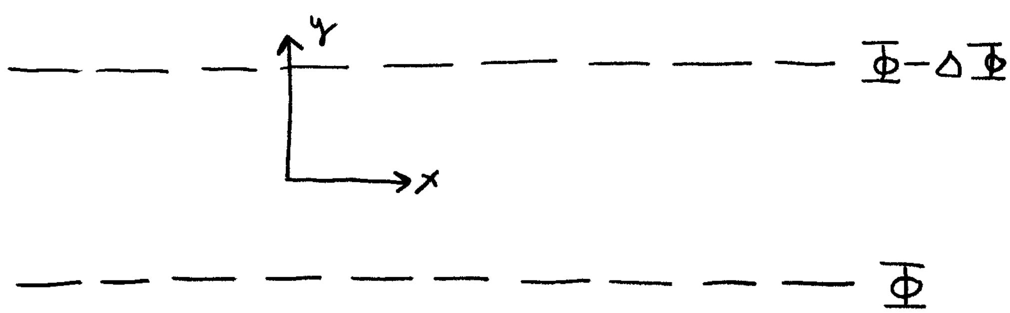

Fig. 4. A natural coordinate system for the horizontal components of the P vector.

Then,

$$P_1 = \partial v_g/\partial x \; \partial\Phi/\partial y \tag{29}$$

$$P_2 = \partial v_g/\partial y \; \partial\Phi/\partial y \tag{30}$$

In the equivalent barotropic wave train shown below (Fig. 5), we can use (29) and (30) to sketch the field of the horizontal component of the P vector, and then determine χ qualitatively. Note that $\partial\Phi/\partial y < 0$, $\partial v_g/\partial y = 0$ along the trough and the ridge axes, and that $\partial v_g/\partial x < 0$ (>0) along the ridge (trough) axis.

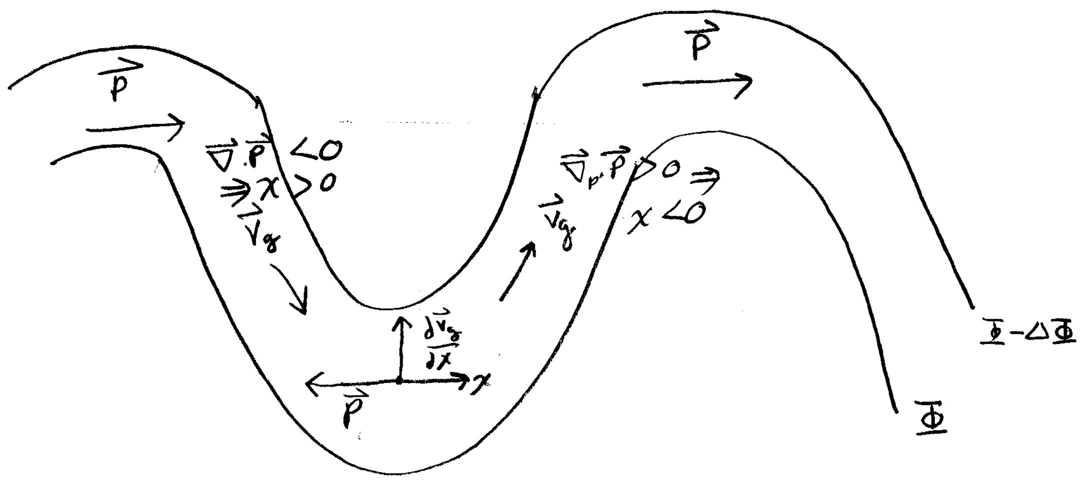

Fig. 5. Illustration of the use of the P-vector to diagnose geopotential-height tendency

So, $\mathbf{P} = - |\partial\Phi/\partial y| (\partial v_g/\partial x\ \mathbf{i} - \partial u_g/\partial x\ \mathbf{j})$, where the equation of continuity has been used to substitute for $\partial v_g/\partial y$. Then the quasi-horizontal part of the P vector

$$\mathbf{P}_\mathbf{p} = |\partial\Phi/\partial y| (\mathbf{k} \times \partial \mathbf{v}_g/\partial x).$$

The quasi-horizontal component of the P vector therefore points in the direction normal and to the left of $\partial \mathbf{v}_g/\partial x$. (Recall that in the natural-coordinate calculation of the Q vector, the latter points in the direction normal to and the *right* of $\partial \mathbf{v}_g/\partial x$. But be careful here! The natural coordinate system in which the Q vector is calculated is defined in terms of the orientation of the isotherms; the natural coordinate system in which the P vector is calculated is defined in terms of the orientation of the geopotential height contours.)

How is the P vector related to the Q vector and the C vector? Differentiating (22) and (23) with respect to p, and (24) with respect to x, and with respect to y, we obtain the following:

$$\partial u_a/\partial p = - 1 / f_o [\ \partial/\partial t\ (\ \partial v_g/\partial p) - 1 / f_o\ \partial P_1/\partial p\ ] - \quad \beta\ y / f_o\ \partial u_g/\partial p \qquad (31)$$

$$\partial v_a/\partial p = \ 1 / f_o [\ \partial/\partial t\ (\ \partial u_g/\partial p) + 1 / f_o\ \partial P_2/\partial p\ ] - \quad \beta\ y / f_o\ \partial v_g/\partial p \qquad (32)$$

$$\partial\omega/\partial x = 1 / \sigma \; [R / p \; \partial/\partial t \, (\partial T/\partial x) + \; \sigma / f_o^2 \; \partial P_3/\partial x \,] \tag{33}$$

$$\partial\omega/\partial y = 1 / \sigma \; [R / p \; \partial/\partial t \, (\partial T/\partial y) + \; \sigma \; / f_o^2 \; \partial P_3/\partial y \,] \tag{34}$$

Using the thermal-wind relation (4.1.121) in (31), we eliminate $\partial/\partial t \, (\partial T/\partial x)$ from (33) and (31), and find that

$$\partial\omega/\partial x - f_o^2/\sigma \; \partial u_a/\partial p = 1 / f_o^2 \; \partial P_3/\partial x - 1/\sigma \; \partial P_1/\partial p + \; \beta \; y / \sigma \; R / p \; \partial T/\partial y \tag{35}$$

Using the thermal-wind relation (4.1.120) in (32), we eliminate $\partial/\partial t \, (\partial T/\partial y)$ from (34) and (32), and find that

$$\partial\omega/\partial y - f_o 2/\sigma \; \partial v_a/\partial p = 1 / f_o^2 \; \partial P_3/\partial y - 1/\sigma \; \partial P_2/\partial p - \; \beta \; y / \sigma \; R / p \; \partial T/\partial x \tag{36}$$

Comparing (35) to (5.7.51), and (36) to (5.7.50), we find that

$$Q_1 = \; 1 / \sigma \; \partial P_1/\partial p - 1 / f_o^2 \; \partial P_3/\partial x \tag{37}$$

$$Q_2 = \; 1 / \sigma \; \partial P_2/\partial p - 1 / f_o^2 \; \partial P_3/\partial y \tag{38}$$

It follows that the Q vector is directed perpendicular and to the right of the horizontal component of the curl of the P vector. In other words, the horizontal component of the C vector is directed in the direction opposite that of the horizontal component of the curl of the P vector.

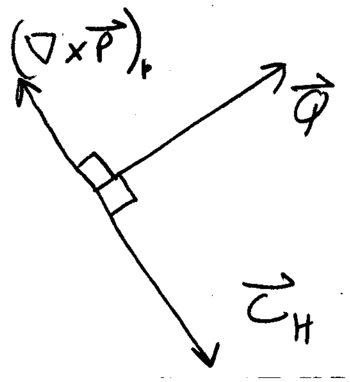

Fig. 6. Relationship among **Q**, $\mathbf{C_H}$, and $\boldsymbol{\nabla_p}$ **X P**.

Differentiating (23) with respect to x, differentiating (22) with respect to y, and subtracting the latter from the former, we find that

$$\zeta_a = 1 / f_o^2 \, (\partial P_2/\partial x - \partial P_1/\partial y) - \beta \, y/f_o \, (\partial v_g/\partial x - \partial u_g/\partial y) + \beta \, y \, u_g / f_o =$$

$$1 / f_o^2 \, \mathbf{k} \bullet \nabla \, \mathbf{X} \, \mathbf{P} \; - \; \beta \, y/f_o \, \zeta_g \; + \; \beta \, y/f_o \, u_g \qquad (39)$$

Thus, the vertical component of the curl of the P-vector field is proportional to the vorticity of the ageostrophic wind. We noted earlier (7) that the vertical component of the C vector is also proportional to the vorticity of the ageostrophic wind. Therefore the vertical component of the C vector is proportional to the vertical component of the curl of the P-vector field. We conclude that the C vector is proportional to the curl of the P-vector field. It remains to be seen whether or not either ever get used operationally and become as in vogue as the Q-vector field is now!

A final thought: The Q, C, and P vectors are the synoptic meteorologist's version of the physicist's unified field theory.