

Advanced Synoptic Meteorology  
METR 5413  
Bluestein

Solutions to Problem Set #6

2.29

(a) Since the atmosphere is barotropic in this problem,

$$\frac{\partial u_g}{\partial p} = 0, \quad \frac{\partial \theta}{\partial y} = 0, \quad \text{and} \quad \frac{\partial \theta}{\partial x} = 0.$$

Therefore, the Sawyer-Eliassen equation is given by

$$\begin{aligned} a \frac{\partial^2 \psi}{\partial y^2} + c \frac{\partial^2 \psi}{\partial p^2} &= -\frac{R}{f_o p} \frac{1}{C_p} \frac{\partial}{\partial y} \left( \frac{dQ}{dt} \right) \\ &= -\frac{A_o}{f_o C_p} \left( \frac{2\pi}{L} \right) \sin\left(\frac{2\pi}{L}y\right) \sin\left(\frac{2\pi}{P_o}p\right) \end{aligned}$$

where

$$a = -\frac{R}{f_o p} \left( \frac{p}{P_o} \right)^k \frac{\partial \theta}{\partial p} \quad \text{and} \quad c = f_o - \frac{\partial u_g}{\partial y}.$$

Try the solution  $\psi = K \sin\left(\frac{2\pi}{L}y\right) \sin\left(\frac{2\pi}{P_o}p\right)$ .

$$\begin{aligned} a \frac{\partial^2 \psi}{\partial y^2} + c \frac{\partial^2 \psi}{\partial p^2} &= -K \left[ a \left( \frac{2\pi}{L} \right)^2 + c \left( \frac{2\pi}{P_o} \right)^2 \right] \sin\left(\frac{2\pi}{L}y\right) \sin\left(\frac{2\pi}{P_o}p\right) \\ &= -\frac{A_o}{f_o C_p} \left( \frac{2\pi}{L} \right) \sin\left(\frac{2\pi}{L}y\right) \sin\left(\frac{2\pi}{P_o}p\right) \end{aligned}$$

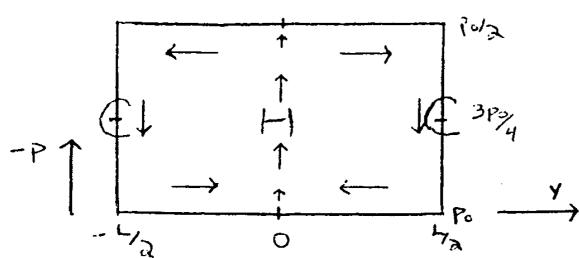
$$\Rightarrow K = \frac{\frac{A_o}{f_o C_p} \left( \frac{2\pi}{L} \right)}{a \left( \frac{2\pi}{L} \right)^2 + c \left( \frac{2\pi}{P_o} \right)^2}$$

$$\text{so, } \psi = \frac{A_o}{f_o C_p} \left( \frac{2\pi}{L} \right) \left[ a \left( \frac{2\pi}{L} \right)^2 + c \left( \frac{2\pi}{P_o} \right)^2 \right]^{-1} \sin\left(\frac{2\pi}{L}y\right) \sin\left(\frac{2\pi}{P_o}p\right)$$

$$v_a = \frac{\partial \psi}{\partial p} = -\frac{A_o}{f_o C_p} \left[ a \left( \frac{2\pi}{L} \right)^2 + c \left( \frac{2\pi}{P_o} \right)^2 \right]^{-1} \left( \frac{2\pi}{L} \right) \left( \frac{2\pi}{P_o} \right) \sin\left(\frac{2\pi}{L}y\right) \cos\left(\frac{2\pi}{P_o}p\right)$$

$$\omega = \frac{\partial \psi}{\partial y} = \frac{A_o}{f_o C_p} \left[ a \left( \frac{2\pi}{L} \right)^2 + c \left( \frac{2\pi}{P_o} \right)^2 \right]^{-1} \left( \frac{2\pi}{L} \right)^2 \cos\left(\frac{2\pi}{L}y\right) \sin\left(\frac{2\pi}{P_o}p\right)$$

$$v_a = 0 \text{ at } y = \pm \frac{L}{2}; \quad \omega = 0 \text{ at } p = P_o, \frac{P_o}{2}$$



(b) In the quasigeostrophic case,  $c = f_o$ , rather than  $f_o - \frac{\partial u_g}{\partial y} > f_o$   
 $\implies |v_a|, |\omega|$  stronger; circulation otherwise identical

## 2.40

frictionless equations of motion subject to GM approximation:

$$\begin{aligned}\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + u_a \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} + v_a \frac{\partial u_g}{\partial y} + \omega \frac{\partial u_g}{\partial p} &= fv_a \\ \frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + u_a \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} + v_a \frac{\partial v_g}{\partial y} + \omega \frac{\partial v_g}{\partial p} &= -fu_a\end{aligned}$$

take  $-\frac{\partial}{\partial y}$  of the first equation and  $\frac{\partial}{\partial x}$  of the second:

$$\begin{aligned}\frac{\partial}{\partial t} \left( \frac{\partial v_g}{\partial x} \right) + \frac{\partial u_g}{\partial x} \frac{\partial v_g}{\partial x} + u_g \frac{\partial}{\partial x} \left( \frac{\partial v_g}{\partial x} \right) + \frac{\partial u_a}{\partial x} \frac{\partial v_g}{\partial x} + u_a \frac{\partial}{\partial x} \left( \frac{\partial v_g}{\partial x} \right) + \frac{\partial v_g}{\partial x} \frac{\partial v_g}{\partial y} + v_g \frac{\partial}{\partial y} \left( \frac{\partial v_g}{\partial x} \right) \\ + \frac{\partial v_a}{\partial x} \frac{\partial v_g}{\partial y} + v_a \frac{\partial}{\partial y} \left( \frac{\partial v_g}{\partial x} \right) + \frac{\partial \omega}{\partial x} \frac{\partial v_g}{\partial p} + \omega \frac{\partial}{\partial p} \left( \frac{\partial v_g}{\partial x} \right) &= -f \frac{\partial u_a}{\partial x}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t} \left( \frac{\partial u_g}{\partial y} \right) - \frac{\partial u_g}{\partial y} \frac{\partial u_g}{\partial x} + u_g \frac{\partial}{\partial x} \left( \frac{\partial u_g}{\partial y} \right) - \frac{\partial u_a}{\partial y} \frac{\partial u_g}{\partial x} + u_a \frac{\partial}{\partial x} \left( \frac{\partial u_g}{\partial y} \right) - \frac{\partial v_g}{\partial y} \frac{\partial u_g}{\partial y} + v_g \frac{\partial}{\partial y} \left( \frac{\partial u_g}{\partial y} \right) \\ - \frac{\partial v_a}{\partial y} \frac{\partial u_g}{\partial y} + v_a \frac{\partial}{\partial y} \left( \frac{\partial u_g}{\partial y} \right) - \frac{\partial \omega}{\partial y} \frac{\partial u_g}{\partial p} + \omega \frac{\partial}{\partial p} \left( \frac{\partial u_g}{\partial y} \right) &= -f \frac{\partial v_a}{\partial y}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t} \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) + u_g \frac{\partial}{\partial x} \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) + v_g \frac{\partial}{\partial y} \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) \\ + u_a \frac{\partial}{\partial x} \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) + v_a \frac{\partial}{\partial y} \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) + \omega \frac{\partial}{\partial p} \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) \\ = -f \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \frac{\partial \omega}{\partial x} \frac{\partial v_g}{\partial p} + \frac{\partial \omega}{\partial y} \frac{\partial u_g}{\partial p} + \frac{\partial u_a}{\partial y} \frac{\partial u_g}{\partial x} + \frac{\partial v_a}{\partial y} \frac{\partial u_g}{\partial y} - \frac{\partial u_a}{\partial x} \frac{\partial v_g}{\partial x} - \frac{\partial v_a}{\partial x} \frac{\partial v_g}{\partial y}\end{aligned}$$

$$\frac{D}{Dt} (\zeta_g + f) = -\delta f + \hat{k} \cdot \frac{\partial \vec{v}_g}{\partial p} \times \vec{\nabla} \omega + \frac{\partial u_g}{\partial x} \frac{\partial u_a}{\partial y} - \frac{\partial v_g}{\partial x} \frac{\partial u_a}{\partial x} + \frac{\partial u_g}{\partial y} \frac{\partial v_a}{\partial y} - \frac{\partial v_g}{\partial y} \frac{\partial v_a}{\partial x}$$

$$\text{but, } -\delta \zeta_g = -\left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) = -\frac{\partial u_a}{\partial x} \frac{\partial v_g}{\partial x} + \frac{\partial u_a}{\partial x} \frac{\partial u_g}{\partial y} - \frac{\partial v_a}{\partial y} \frac{\partial v_g}{\partial x} + \frac{\partial v_a}{\partial y} \frac{\partial u_g}{\partial y}$$

$$\text{therefore, } \frac{D}{Dt} (\zeta_g + f) = -\delta (\zeta_g + f) + \hat{k} \cdot \frac{\partial \vec{v}_g}{\partial p} \times \vec{\nabla} \omega + \frac{\partial u_g}{\partial x} \frac{\partial u_a}{\partial y} - \frac{\partial v_g}{\partial y} \frac{\partial v_a}{\partial x} - \frac{\partial u_g}{\partial y} \frac{\partial u_a}{\partial x} + \frac{\partial v_g}{\partial x} \frac{\partial v_a}{\partial y}$$

$$\begin{aligned}
& \frac{\partial u_g}{\partial x} \frac{\partial u_a}{\partial y} - \frac{\partial v_g}{\partial y} \frac{\partial v_a}{\partial x} - \frac{\partial u_g}{\partial y} \frac{\partial u_a}{\partial x} + \frac{\partial v_g}{\partial x} \frac{\partial v_a}{\partial y} = -\frac{\partial v_g}{\partial y} \frac{\partial u_a}{\partial y} + \frac{\partial u_g}{\partial x} \frac{\partial v_a}{\partial x} - \frac{\partial u_g}{\partial y} \frac{\partial u_a}{\partial x} + \frac{\partial v_g}{\partial x} \frac{\partial v_a}{\partial y} \\
&= -\frac{\partial}{\partial y} \left( v_g \frac{\partial u_a}{\partial y} \right) + v_g \frac{\partial}{\partial y} \left( \frac{\partial u_a}{\partial y} \right) + \frac{\partial}{\partial x} \left( u_g \frac{\partial v_a}{\partial x} \right) - u_g \frac{\partial}{\partial x} \left( \frac{\partial v_a}{\partial x} \right) \\
&\quad - \frac{\partial}{\partial y} \left( u_g \frac{\partial u_a}{\partial x} \right) + u_g \frac{\partial}{\partial y} \left( \frac{\partial u_a}{\partial x} \right) + \frac{\partial}{\partial x} \left( v_g \frac{\partial v_a}{\partial y} \right) - v_g \frac{\partial}{\partial x} \left( \frac{\partial v_a}{\partial y} \right) \\
&= -\frac{\partial}{\partial y} \left( u_g \frac{\partial u_a}{\partial x} + v_g \frac{\partial u_a}{\partial y} \right) + \frac{\partial}{\partial x} \left( u_g \frac{\partial v_a}{\partial x} + v_g \frac{\partial v_a}{\partial y} \right) - u_g \frac{\partial}{\partial x} \left( \frac{\partial v_a}{\partial x} - \frac{\partial u_a}{\partial y} \right) - v_g \frac{\partial}{\partial y} \left( \frac{\partial v_a}{\partial x} - \frac{\partial u_a}{\partial y} \right) \\
&= -\frac{\partial}{\partial y} \left( u_g \frac{\partial u_a}{\partial x} + v_g \frac{\partial u_a}{\partial y} \right) + \frac{\partial}{\partial x} \left( u_g \frac{\partial v_a}{\partial x} + v_g \frac{\partial v_a}{\partial y} \right) - \vec{v}_g \cdot \vec{\nabla} \zeta_a
\end{aligned}$$

If we make the geostrophic momentum approximation, then

$$u_g \frac{\partial u_a}{\partial x} + v_g \frac{\partial u_a}{\partial y} = 0 \quad \text{and} \quad u_g \frac{\partial v_a}{\partial x} + v_g \frac{\partial v_a}{\partial y} = 0.$$

Furthermore, if we neglect the geostrophic advection of ageostrophic vorticity, then

$$\frac{\partial u_g}{\partial x} \frac{\partial u_a}{\partial y} - \frac{\partial v_g}{\partial y} \frac{\partial v_a}{\partial x} - \frac{\partial u_g}{\partial y} \frac{\partial u_a}{\partial x} + \frac{\partial v_g}{\partial x} \frac{\partial v_a}{\partial y} = 0.$$

$$\implies \frac{D}{Dt} (\zeta_g + f) = -\delta(\zeta_g + f) + \hat{k} \cdot \frac{\partial \vec{v}_g}{\partial p} \times \vec{\nabla} \omega$$

2.28

$$\frac{DX}{Dt} = u_a$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi}{\partial y} = -fu + fu_g = -f(u - u_g) = -fu_a$$

$$u_a = -\frac{1}{f} \frac{Dv}{Dt} = -\frac{1}{f} \frac{D^2y}{Dt^2}$$

$$\frac{DX}{Dt} = -\frac{1}{f} \frac{D^2y}{Dt^2}$$

$$X = -\frac{1}{f} \frac{Dy}{Dt} \quad (= -\frac{v}{f})$$

2.42

$$\vec{v}_a = \frac{1}{f} \hat{k} \times \frac{D\vec{v}}{Dt}$$

$$\begin{aligned}\vec{v} &= \vec{v}_g + \frac{1}{f} \hat{k} \times \frac{D\vec{v}}{Dt} \\ &= \vec{v}_g + \frac{1}{f} \hat{k} \times \frac{D}{Dt} \left( \vec{v}_g + \frac{1}{f} \hat{k} \times \frac{D\vec{v}}{Dt} \right) \\ &= \vec{v}_g + \frac{1}{f} \hat{k} \times \frac{D\vec{v}_g}{Dt} - \frac{1}{f^2} \frac{D^2 \vec{v}}{Dt^2} \\ &= \vec{v}_g + \frac{1}{f} \hat{k} \times \frac{D\vec{v}_g}{Dt} - \frac{1}{f^2} \frac{D^2 \vec{v}_g}{Dt^2} \left( \vec{v}_g + \frac{1}{f} \hat{k} \times \frac{D\vec{v}_g}{Dt} - \frac{1}{f^2} \frac{D^2 \vec{v}}{Dt^2} \right) \\ &= \vec{v}_g + \frac{1}{f} \hat{k} \times \frac{D\vec{v}_g}{Dt} - \frac{1}{f^2} \frac{D^2 \vec{v}_g}{Dt^2} - \frac{1}{f^3} \hat{k} \times \frac{D^3 \vec{v}_g}{Dt^3} + \frac{1}{f^4} \frac{D^4 \vec{v}}{Dt^4}\end{aligned}$$

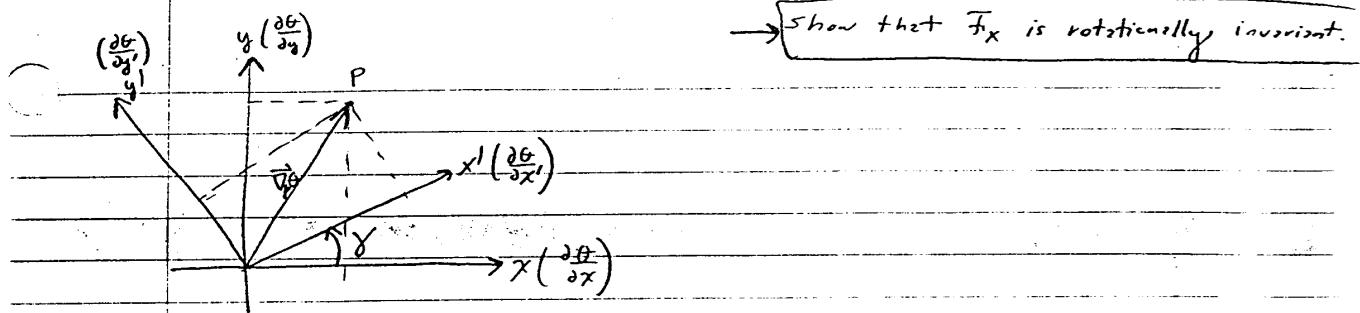
$$\begin{aligned}\text{so, } \vec{v} - \vec{v}_g &= \vec{v}_a = \frac{1}{f} \hat{k} \times \frac{D\vec{v}_g}{Dt} - \frac{1}{f^2} \frac{D^2 \vec{v}_g}{Dt^2} - \frac{1}{f^3} \hat{k} \times \frac{D^3 \vec{v}_g}{Dt^3} \\ &\quad + \frac{1}{f^4} \frac{D^4 \vec{v}}{Dt^4} \left( \vec{v}_g + \frac{1}{f} \hat{k} \times \frac{D\vec{v}_g}{Dt} - \frac{1}{f^2} \frac{D^2 \vec{v}_g}{Dt^2} - \frac{1}{f^3} \hat{k} \times \frac{D^3 \vec{v}_g}{Dt^3} + \frac{1}{f^4} \frac{D^4 \vec{v}}{Dt^4} \right) \\ &= \frac{1}{f} \hat{k} \times \frac{D\vec{v}_g}{Dt} - \frac{1}{f^3} \hat{k} \times \frac{D^3 \vec{v}_g}{Dt^3} + \frac{1}{f^5} \hat{k} \times \frac{D^5 \vec{v}_g}{Dt^5} - \frac{1}{f^7} \hat{k} \times \frac{D^7 \vec{v}_g}{Dt^7} \\ &\quad - \frac{1}{f^2} \frac{D^2 \vec{v}_g}{Dt^2} + \frac{1}{f^4} \frac{D^4 \vec{v}_g}{Dt^4} - \frac{1}{f^6} \frac{D^6 \vec{v}_g}{Dt^6} + \frac{1}{f^8} \frac{D^8 \vec{v}}{Dt^8} \\ &= \dots\end{aligned}$$

therefore, by induction

$$\vec{v}_a = \sum_{i=1}^{n/2} (-1)^{i-1} \frac{1}{f^{2i-1}} \hat{k} \times \frac{D^{2i-1} \vec{v}_g}{Dt^{2i-1}} + \sum_{i=1}^{n/2-1} (-1)^i \frac{1}{f^{2i}} \frac{D^{2i} \vec{v}_g}{Dt^{2i}} + (-1)^{n/2} \frac{1}{f^n} \frac{D^n \vec{v}}{Dt^n}$$

for n=2, 4, 6, . . .

$$F_x = \frac{1}{|\nabla_p|} \left[ -\left(\frac{\partial \theta}{\partial x}\right)^2 \frac{\partial u}{\partial y} - \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \frac{\partial u}{\partial x} + \left(\frac{\partial \theta}{\partial y}\right)^2 \frac{\partial v}{\partial x} \right] \quad (1)$$



Vol. I  
(3.1.15)

$$\begin{pmatrix} \frac{\partial u'}{\partial x'} & \frac{\partial u'}{\partial y'} \\ \frac{\partial v'}{\partial x'} & \frac{\partial v'}{\partial y'} \end{pmatrix} = \begin{pmatrix} \cos \gamma \frac{\partial u}{\partial x} + \sin \gamma \frac{\partial v}{\partial x} & \cos \gamma \frac{\partial u}{\partial y} + \sin \gamma \frac{\partial v}{\partial y} \\ -\sin \gamma \frac{\partial u}{\partial x} + \cos \gamma \frac{\partial v}{\partial x} & -\sin \gamma \frac{\partial u}{\partial y} + \cos \gamma \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix}$$

$$\frac{\partial u'}{\partial x'} = \cos^2 \gamma \frac{\partial u}{\partial x} + \cos \gamma \sin \gamma \frac{\partial v}{\partial x} + \cos \gamma \sin \gamma \frac{\partial u}{\partial y} + \sin^2 \gamma \frac{\partial v}{\partial y}$$

$$\frac{\partial u'}{\partial y'} = -\sin \gamma \cos \gamma \frac{\partial u}{\partial x} - \sin^2 \gamma \frac{\partial v}{\partial x} + \cos^2 \gamma \frac{\partial u}{\partial y} + \cos \gamma \sin \gamma \frac{\partial v}{\partial y}$$

$$\frac{\partial v'}{\partial x'} = -\sin \gamma \cos \gamma \frac{\partial v}{\partial x} + \cos^2 \gamma \frac{\partial v}{\partial y} - \sin^2 \gamma \frac{\partial u}{\partial y} + \cos \gamma \sin \gamma \frac{\partial u}{\partial y}$$

$$\frac{\partial v'}{\partial y'} = \sin^2 \gamma \frac{\partial u}{\partial x} + \sin \gamma \cos \gamma \frac{\partial v}{\partial x} - \sin \gamma \cos \gamma \frac{\partial u}{\partial y} + \cos^2 \gamma \frac{\partial v}{\partial y}$$

$$\frac{\partial \theta}{\partial x'} = \cos \gamma \frac{\partial \theta}{\partial x} + \sin \gamma \frac{\partial \theta}{\partial y}$$

$$\cos(\widehat{\gamma} + \gamma) = \frac{\cos \gamma}{\cos \gamma - \sin^2 \gamma}$$

$$\frac{\partial \theta}{\partial y'} = -\sin \gamma \frac{\partial \theta}{\partial x} + \cos \gamma \frac{\partial \theta}{\partial y}$$

$$-\left(\frac{\partial \theta}{\partial x}\right)^2 = -\cos^2 \gamma \left(\frac{\partial \theta}{\partial x}\right)^2 - 2 \sin \gamma \cos \gamma \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} - \sin^2 \gamma \left(\frac{\partial \theta}{\partial y}\right)^2$$

$$-\frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y'} = -\cos \gamma \sin \gamma \left(\frac{\partial \theta}{\partial x}\right)^2 + \underbrace{\cos^2 \gamma \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y}}_{\cos 2\gamma \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y}} - \sin^2 \gamma \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} + \sin \gamma \cos \gamma \left(\frac{\partial \theta}{\partial y}\right)^2$$

$$-\frac{\partial \theta}{\partial x'} \frac{\partial \theta}{\partial y} = \cos \gamma \sin \gamma \left(\frac{\partial \theta}{\partial x}\right)^2 - \underbrace{\cos^2 \gamma \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y}}_{-\cos 2\gamma \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y}} + \sin^2 \gamma \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} - \sin \gamma \cos \gamma \left(\frac{\partial \theta}{\partial y}\right)^2$$

$$\left(\frac{\partial \theta}{\partial y'}\right)^2 = \sin^2 \gamma \left(\frac{\partial \theta}{\partial x}\right)^2 - 2 \sin \gamma \cos \gamma \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} + \cos^2 \gamma \left(\frac{\partial \theta}{\partial y}\right)^2$$

$$\left(\frac{\partial \theta}{\partial x}\right)^2 \frac{\partial v}{\partial y} \left[ \cos^4 \gamma + 2 \sin^2 \gamma \cos^2 \gamma + \sin^4 \gamma \right]$$

$$(\sin^2 \gamma + \cos^2 \gamma)^2$$

$$-\frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \frac{\partial v}{\partial y} \left[ 2 \sin^2 \gamma \cos^2 \gamma + \cos^2 \gamma \cos^2 \gamma - \cos^2 \gamma \sin^2 \gamma + 2 \sin^2 \gamma \cos^2 \gamma \right]$$

$$\cos^4 \gamma - \sin^2 \gamma \cos^2 \gamma - \cos^2 \gamma \sin^2 \gamma + \sin^4 \gamma$$

$$(\sin^2 \gamma + \cos^2 \gamma)^2$$

$$-\frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \frac{\partial v}{\partial x} \left[ 2 \sin^2 \gamma \cos^2 \gamma - \cos^2 \gamma \sin^2 \gamma + \cos^2 \gamma \cos^2 \gamma + 2 \sin^2 \gamma \cos^2 \gamma \right]$$

$$-\sin^2 \gamma \cos^2 \gamma + \sin^4 \gamma + \cos^4 \gamma - \sin^2 \gamma \cos^2 \gamma$$

$$(\sin^2 \gamma + \cos^2 \gamma)^2$$

$$\left(\frac{\partial \theta}{\partial y}\right)^2 \frac{\partial v}{\partial x} \left[ \sin^4 \gamma + 2 \sin^2 \gamma \cos^2 \gamma + \cos^4 \gamma \right]$$

$$(\sin^2 \gamma + \cos^2 \gamma)^2$$



$$\frac{\partial V}{\partial x} \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \left[ 2 \sin^3 \gamma \cos \gamma + 2 \cos^2 \gamma \sin \gamma \cos \gamma - 2 \sin \gamma \cos^3 \gamma \right]$$

$$2(\cos^3 \gamma \sin \gamma - \sin^3 \gamma \cos \gamma)$$

$$\cancel{\sin^3 \gamma (2 \cos \gamma - 2 \cos \gamma)}_0 + \cos^3 \gamma (\cancel{2 \sin \gamma - 2 \sin \gamma})_0 = 0$$

$$\frac{\partial V}{\partial y} \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \left[ -2 \sin \gamma \cos^3 \gamma + 2 \cos^2 \gamma \sin \gamma \cos \gamma + 2 \sin^3 \gamma \cos \gamma \right]$$

$$2 \cos^3 \gamma \sin \gamma - 2 \sin^3 \gamma \cos \gamma$$

$$\cancel{\sin^3 \gamma (-2 \cos \gamma + 2 \cos \gamma)}_0 + \cos^3 \gamma (\cancel{2 \sin \gamma - 2 \sin \gamma})_0 = 0$$

$$\therefore \vec{F}_x = \frac{1}{|\vec{\nabla}_r \theta|} \left[ -\left( \frac{\partial \theta}{\partial x'} \right)^2 \frac{\partial V'}{\partial y'} - \frac{\partial \theta}{\partial x'} \frac{\partial \theta}{\partial y'} \frac{\partial V'}{\partial y'} + \frac{\partial \theta}{\partial x'} \frac{\partial \theta}{\partial y'} \frac{\partial V'}{\partial x'} + \left( \frac{\partial \theta}{\partial y'} \right)^2 \frac{\partial V'}{\partial x'} \right]$$

$$= \frac{1}{|\vec{\nabla}_r \theta|} \left[ -\left( \frac{\partial \theta}{\partial x} \right)^2 \frac{\partial V}{\partial y} - \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \frac{\partial V}{\partial y} + \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \frac{\partial V}{\partial x} + \left( \frac{\partial \theta}{\partial y} \right)^2 \frac{\partial V}{\partial x} \right]$$

✓

$\vec{F}_x$  is rotationally invariant