H. Bluestein notes for METR 6223

Derivation of (2.281) – (2.285)

①

10/7/14

$$\bar{U} = \frac{U_0}{H} z$$

$$+ w\frac{d\bar{U}}{dz}$$

u-eqn. $$\left(\frac{\partial}{\partial t} + \bar{U}(z)\frac{\partial}{\partial x} - \nu\nabla^2\right)u = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

nondim $O(1)$

scale $$x^* = Hx \quad y^* = Hy \quad z^* = Hz \quad t^* = \frac{H^2}{\nu}t$$

$$\nu\nabla^2(\ ) = \frac{\partial u}{\partial t}$$
$$\frac{\nu(U)}{H^2} \sim \frac{U}{\tau} \Rightarrow \tau \sim \frac{H^2}{\nu}$$

scale $$U^* = \frac{\text{space}}{\text{time}} = \frac{H}{H^2/\nu}\,U = \frac{\nu}{H}U \qquad V^* = \frac{\nu}{H}V \qquad W^* = \frac{\nu}{H}W$$

drop $()^*$ notation:

$$+\frac{\nu}{H}w\frac{U_0}{H}$$

rename p

$$\left(\frac{\nu}{H^2}\frac{\partial}{\partial t} + \frac{U_0}{\cancel{H}}\cancel{H}z\frac{1}{H}\frac{\partial}{\partial x} - \frac{\nu}{H^2}\nabla^2\right)\frac{\nu}{H}u = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right) = -\frac{1}{H}\frac{\partial(p(\ ))}{\partial x}$$

$$\frac{H^3}{\nu^2}\times$$

$$\left(\frac{\partial}{\partial t} + \underbrace{\left(\frac{U_0 H}{\nu}\right)}_{\equiv Re}z\frac{\partial}{\partial x} - \nabla^2\right)u = -\frac{H^2}{\nu^2}\frac{\partial}{\partial x}\{p(\ )\} \Rightarrow \text{let}$$

$$+ w\left(\frac{U_0 H}{\nu}\right) = Re \qquad p^* = \frac{\nu^2}{H^2}p$$

$$= -\frac{\partial p}{\partial x}$$

so, $$\boxed{\left(\frac{\partial}{\partial t} + Re\,z\frac{\partial}{\partial x} - \nabla^2\right)u = -\frac{\partial p}{\partial x} - Re\,w}$$

v-eqn. $$\boxed{\left(\frac{\partial}{\partial t} + Re\,z\frac{\partial}{\partial x} - \nabla^2\right)v = -\frac{\partial p}{\partial y}}$$

w-eqn: $$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x} - \nu\nabla^2\right)w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + B$$

$$\left(\frac{\nu}{H^2}\frac{\partial}{\partial t} + \frac{U_0}{\cancel{H}}\cancel{H}z\frac{1}{H}\frac{\partial}{\partial x} - \frac{\nu}{H^2}\nabla^2\right)\frac{\nu}{H}w = -\frac{\nu^2}{H^2}\frac{1}{H}\frac{\partial p}{\partial z} + B$$

$$\frac{H^3}{\nu^2}\times$$

$$\left(\frac{\partial}{\partial t} + \underbrace{\left(\frac{U_0 H}{\nu}\right)}_{Re}z\frac{\partial}{\partial x} - \nabla^2\right)w = -\frac{\partial p}{\partial z} + \frac{BH^3}{\nu^2}$$

nondim. $O(1)$

$$B = (\quad)B$$

thermo. eq.: $\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - K\nabla^2\right)B = \alpha w$

$$\left(\frac{\nu}{H^2}\frac{\partial}{\partial t} + \frac{U_0}{\cancel{H}}\cancel{H}z\frac{1}{H}\frac{\partial}{\partial x} - \frac{K\nabla^2}{H^2}\right)B = \alpha\frac{\nu}{H}w$$

$$\times\frac{H^2}{\nu} \qquad \left(\frac{\partial}{\partial t} + \underbrace{\left(\frac{U_0 H}{\nu}\right)}_{Re} z\frac{\partial}{\partial x} - \frac{K\nabla^2}{\nu}\right)B = \alpha H w$$

$$\left(\frac{\partial}{\partial t} + Re\, z\frac{\partial}{\partial x}\right)B - \frac{1}{\sigma}\nabla^2 B = \alpha H w$$

$$\sigma\left(\frac{\partial}{\partial t} + Re\, z\frac{\partial}{\partial x}\right)B - \nabla^2 B = \alpha H w\frac{\nu}{K}$$

try $B = (\alpha H\sigma)B \Rightarrow$

$$\sigma\left(\frac{\partial}{\partial t} + Re\, z\frac{\partial}{\partial x}\right)B(\alpha H\sigma) - \nabla^2\left[B(\alpha H\sigma)\right] = \left(\frac{\alpha H\nu}{K}\right)w$$

$$\boxed{\sigma\left(\frac{\partial}{\partial t} + Re\, z\frac{\partial}{\partial x}\right)B - \nabla^2 B = \frac{\cancel{\alpha H\nu/K}}{\cancel{\alpha H\nu/K}}\, w = w}$$

so, vert. eqn:

$$\left(\frac{\partial}{\partial t} + Re\, z\frac{\partial}{\partial x} - \nabla^2\right)w = -\frac{\partial p}{\partial z} + B\underbrace{\frac{H^3}{\nu^2}(\alpha H\sigma)}_{\underbrace{\frac{\alpha H^4\nu}{\nu^2 K}}_{\frac{\alpha H^4}{\nu K} = Ra}}$$

$$\boxed{\left(\frac{\partial}{\partial t} + Re\, z\frac{\partial}{\partial x} - \nabla^2\right)w = -\frac{\partial p}{\partial z} + B\, Ra}$$

$$\cancel{\frac{1}{H}}\frac{\partial U}{\partial x}\cancel{\frac{\nu}{H}} + \cancel{\frac{1}{H}}\frac{\partial v}{\partial y}\cancel{\frac{\nu}{H}} + \cancel{\frac{1}{H}}\frac{\partial w}{\partial z}\cancel{\frac{\nu}{H}} = 0$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$