METR 6223

Fall 2021

Solutions to Exam #1

1 (a) $\frac{BH^3}{\nu K} \equiv Ra$, where H is vertical spacing between upper & lower boundaries, $B \equiv$ buoyancy

Ra_c is smallest Ra for which stationary overturning can occur.

(b) $\frac{D\vec{v}}{Dt} = \cancel{-\frac{1}{\rho}\vec{\nabla}p} + B + \nu\nabla^2\vec{v}$

neglect for narrow parcel

$Re = \frac{\left|\frac{D\vec{v}}{Dt}\right|}{|\nu\nabla^2\vec{v}|} \gg 1 \Rightarrow \frac{D\vec{v}}{Dt} \sim B$

$\Rightarrow B \sim U/(L/U)$

$\Rightarrow B \sim \frac{U^2}{L}$

$(U \sim W$, $L \sim H)$

(c) $T_0 = \frac{|\Omega \times \vec{v}|}{\nu\nabla^2\vec{v}} \sim \frac{\Omega U}{\nu U/L^2} = \frac{\Omega L^2}{\nu}$

$\equiv \frac{|\text{Coriolis force}|}{|\text{molecular friction force}|}$

large $T_0 \Rightarrow$ rotation stabilizes Rayleigh–Bénard convection

i.e. Ra_c increases w/ T_0

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density in density current

$$c = \sqrt{2gh\,\frac{\rho_0 - \rho_1}{\rho_1}}$$

depth of density current

density of ambient air

$$p = \rho RT \Rightarrow \rho = \frac{p}{RT} \sim \frac{1000\,hPa}{R\,T_{sfc}}$$

$$\rho_1 = \frac{1000\,\cancel{hPa}}{(287\,m^2 s^{-2} K^{-1})(\underbrace{273.15+30}_{303.15\,K})} \times \frac{kg\,m s^{-2} m^{-2}}{1\,Pa} \times \frac{10^5\,Pa}{1000\,hPa}$$

$$= 1.149\,kg\,m^{-3}$$

$$\rho_0 \approx \frac{1000\,hPa}{(287\,m^2 s^{-2} K^{-1})(\underbrace{303.15-7.5}_{295.7})K} \times \frac{kg\,m s^{-2} m^{-2}}{1\,Pa} \times \frac{10^5\,Pa}{1000\,hPa}$$

$$= 1.178\,kg\,m^{-3}$$

(a)

$$c = \sqrt{2(9.8\,m s^{-2})(10^3\,m)\left(\frac{1.178 - 1.149}{1.149}\right)}$$

$$= \boxed{22.2\,m s^{-1}}$$

(b) $\cancel{\frac{\partial U}{\partial t}}^{\text{steady}} + U\frac{\partial U}{\partial x} + \cancel{V\frac{\partial U}{\partial y}}^{\text{2-D}} + \cancel{W\frac{\partial U}{\partial z}}^{W=0 \text{ at } z=0} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \cancel{\nu\nabla^2 U}^{\text{frictionless}}$

$$\frac{\partial}{\partial x}\left(\frac{1}{2}U^2 + \frac{p'}{\rho}\right) = 0$$

$$\frac{1}{2}U^2 + \frac{p'}{\rho} = \text{const.}$$

well inside density current $U = 22.2\ ms^{-1}$

at edge of density current $U = 0\ ms^{-1}$

well inside density current $p' = 0$

at edge of density current $\frac{p'}{\rho} = \frac{p'}{1.149\ kg\,m^{-3}}$

$$\Rightarrow \frac{1}{2}(22.2\,ms^{-1})^2 + \cancel{\frac{(0\,Pa)}{\rho}}^{0} = \cancel{\frac{1}{2}(0\,ms^{-1})^2}^{0} + \frac{p'}{1.149\ kg\,m^{-3}}$$

$$\Rightarrow p' = (1.149\ kg\,m^{-3})\frac{1}{2}(22.2\,ms^{-1})^2$$

$$= 283.1\ \underbrace{kg\,\underbrace{m^{-1}s^{-2}}_{ms^{-2}m^{-2}}} = 283.1\ Pa \times \frac{\overbrace{1000}^{10^3}hPa}{10^5\ Pa}$$

$$= \boxed{2.83\ hPa}$$

(c) $\frac{Dw}{Dt} \approx -\frac{1}{\rho}\frac{\partial p'}{\partial z} \approx -\frac{1}{1.149\,kg\,m^{-3}}\left(-\frac{2.83\,hPa}{10^3\,m}\right) \times \frac{10^5\,Pa}{10^3\,hPa} \times \frac{kg\,ms^{-2}\,m^{-2}}{1\,Pa}$

$$= \boxed{+0.246\ m\,s^{-2}}$$

3 (a) shallow atmosphere, no sound waves permitted

$(U \ll C_s)$ ← speed sound

vert scale → $D \ll H$ ← depth of atmos.

(b) deep atmosphere, no sound waves permitted

$\frac{D}{H} \sim 1$ $(U \ll C_s)$

(c) sound waves permitted, $U \gtrsim C_s$

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$$-\delta = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \sim s^{-1} \sim F^a z^b$$

$$F \sim \text{flux of buoyancy} \sim \overset{B}{(ms^{-2})}\overset{\text{area}}{(m^2)}\overset{W}{(ms^{-1})} \sim m^4 s^{-3}$$

$$\Rightarrow s^{-1} \sim (m^4 s^{-3})^a (m)^b$$

$$-1 = -3a \Rightarrow a = 1/3$$

$$0 = 4a + b$$

$$\Rightarrow b = -4(1/3) = -4/3$$

$$\text{so, } -\delta \sim F^{1/3} z^{-4/3}$$

5 for a spherical bubble that does not entrain air from outside the bubble, $\frac{Dw}{Dt} < \underset{B>0}{B}$, owing to $-\frac{1}{\rho}\frac{\partial p'}{\partial z} < 0$ ⇒ overestimate

if there is water vapor, B will increase slightly, still overestimate probably

if there is water or ice suspended in bubble, there will be less B ⇒ overestimate

entrainment will reduce B

never an underestimate!