

$$\left\{ \begin{array}{l} \frac{Dv}{Dt} = -fv - \frac{1}{\rho} \frac{\partial p}{\partial y} = -f(v - v_g) = -f(m - m_g) \\ \frac{Dw}{Dt} = \frac{\partial}{\partial} (\theta' - \theta) \end{array} \right. \text{ buoyancy}$$

See Fig.

assume
no mixing!

like parcel theory, but startwise ascent can
lead to cooler or warmer air plift
than cold!

$$\begin{aligned} mg &= \frac{v_0}{\rho} \frac{\partial \rho}{\partial \theta} \theta' = \left(\frac{v_0}{\rho} \right) \frac{\partial \theta}{\partial \theta} \theta' \\ \frac{\partial \theta}{\partial \theta} &= \frac{v_0}{\rho g} \end{aligned}$$

consider

(3)

$$m\ddot{y} = \dot{m}_g - f_y \quad \text{Eq (2.2.2)}$$

(see Fig.)



$$\frac{\partial m_g}{\partial \theta} = \frac{\partial U_g}{\partial \theta}$$

$$\text{mg-stc.} \quad \dot{m}_g = 0 = \frac{\partial m_g}{\partial y} \dot{y} + \frac{\partial m_g}{\partial \theta} \dot{\theta}$$

Slope of
mg-stc.

$$\left(\frac{dy}{dx} \right)_{m_g} = - \frac{\frac{\partial m_g}{\partial y}}{\frac{\partial m_g}{\partial \theta}}$$

$$f - \frac{\partial U_g}{\partial y} = \frac{S_g + f}{\frac{\partial U_g}{\partial \theta}}$$

(5)

$$\text{recall } \left(\frac{dy}{dx} \right)_0 = \frac{f \frac{\partial U_g}{\partial y}}{\frac{g \frac{\partial \theta}{\partial y}}{\theta \frac{\partial \theta}{\partial y}}}$$

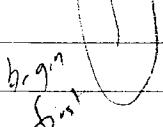
$$\text{so, } \left(\frac{dy}{dx} \right)_0 = \frac{f \frac{\partial U_g}{\partial y}}{\frac{g \frac{\partial \theta}{\partial y}}{\theta \frac{\partial \theta}{\partial y}}} = \frac{\frac{\partial U_g}{\partial y}}{\frac{g \frac{\partial \theta}{\partial y}}{\theta \frac{\partial \theta}{\partial y}}} = \frac{f}{R_1 (S_g + f)}$$

unstable when $R_1 < \frac{f}{S_g + f} \Rightarrow 1 < \frac{f}{R_1 (S_g + f)}$

$$\Rightarrow \frac{\left(\frac{dy}{dx} \right)_0}{\left(\frac{dy}{dx} \right)_{m_g}} > 1$$

Kang Ennai

Slope of S stc. \gg Slope of m_g stc.



let $m = v - f_y \quad \rightarrow \frac{dy}{dt} = 0$

$$\frac{Dm}{Dt} = \frac{Dv}{Dt} - fv = -\frac{1}{\rho} \frac{\partial P}{\partial x} = +v_g = 0 \quad \text{consid!}$$

S10

thermal wind

$$\frac{\partial v_0}{\partial z} = -\frac{g}{fT} \left(\frac{\partial T}{\partial y} \right)_z + \underbrace{\frac{v_0 \frac{\partial T}{\partial z}}{fT}}_{\text{small}} \approx -\frac{g}{fT} \left(\frac{\partial T}{\partial y} \right)_z$$

$$= -\frac{g}{f} \frac{1}{\theta} \left(\frac{\partial \theta}{\partial y} \right)_z$$

along θ site

$$\theta = \theta(y, z) \rightarrow d\theta = 0 = \frac{\partial \theta}{\partial y} dy + \frac{\partial \theta}{\partial z} dz$$

(4)

slope of θ site

$$\left(\frac{d\theta}{dy} \right)_z = \frac{\frac{\partial \theta}{\partial y}}{\frac{\partial \theta}{\partial z}} = \frac{f \frac{\partial u_g}{\partial z}}{g \frac{\partial \theta}{\partial z}}$$

then $\frac{\partial v}{\partial z} = f \left[\frac{\partial u_g}{\partial z} + f \frac{\partial u_g}{\partial z} (\delta y)_0 - \left(f - \frac{\partial u_g}{\partial z} \right) \left(\frac{\partial \theta}{\partial z} \right)_0 \right]$

for displacement along θ -site
 $\frac{D^2(\delta y)_0}{Dt^2} + f \left[\left(f - \frac{\partial u_g}{\partial z} \right) - f \left(\frac{\partial u_g}{\partial z} \right)^2 \right] \left(\frac{\partial \theta}{\partial z} \right)_0$
 $(\delta y)_0$ inst. $\frac{f}{R_i}$

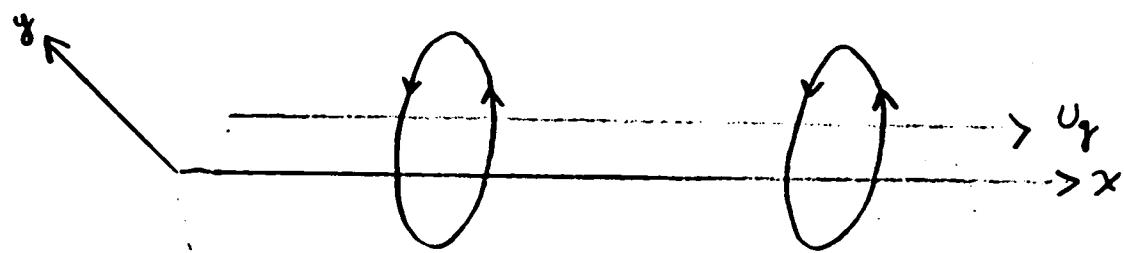
$$\frac{D^2}{Dt^2} (\delta y)_0 + f^2 \left[\frac{s_g + f}{f} - \frac{1}{R_i} \right] (\delta y)_0 = 0$$

$$(\delta y)_0 \sim e^{i\omega t}$$

$$\sigma^2 = f^2 \left[\frac{s_g + f}{f} - \frac{1}{R_i} \right]$$

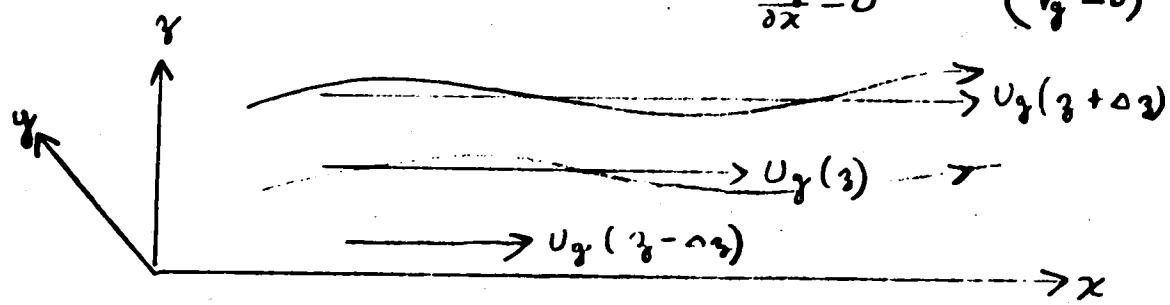
instab $\Rightarrow \sigma^2 < 0 \Rightarrow R_i < \frac{f}{s_g + f}$

for syn. mat.



Symmetrically unstable circulations in
zonal flow

$$\frac{\partial U_g}{\partial x} = 0 \quad (V_g = 0) \quad \frac{\partial U_g}{\partial y} \neq 0$$



growing waves in a baroclinically
unstable zonal flow

$$\frac{\partial U_g}{\partial z} \neq 0 \quad \frac{\partial V_g}{\partial x} \neq 0$$

St

4/21/87 symmetric instability

showed earlier:

Sawyer Eliassen eqn. not elliptic if

$$\left(\frac{\partial u_g}{\partial p}\right)^2 + \left[\frac{R}{\rho_0 p} \left(\frac{p}{p_0}\right)^k\right] \frac{\partial p}{\partial r} \left(\frac{v_0 - \delta u_g}{\delta y}\right) > 0$$

 \equiv nec. condx. for symm. instab.

{ centrifugal
 dynamics
 inertial

1966 Stone \rightarrow symmetric

formal analysis first done by Eady 1949

Emmanuel \rightarrow perturb analysis $\beta_x = 0$

relation to baroclinic instability. (see Fig.)

{ convection \uparrow p undist \rightarrow displaced & evaluates restoring forces
 symm. tube p undist (see Fig.)

$$\text{for tube } \frac{Dv}{Dt} = -fv - \frac{1}{\rho} \frac{\partial p}{\partial y} = f(v_y - u) \quad v_y = v_g(y, z)$$

$$= f \left[\left(U_{g0} + \frac{\partial u_g}{\partial z} \delta z + \frac{\partial u_g}{\partial y} \delta y \right) - u \right]$$

$$z + y, z = 0 \\ \text{small } \delta z, \delta y$$

$$\frac{Dv}{Dt} = fv - \frac{1}{\rho} \frac{\partial p}{\partial x} = f(v - v_g) = fv = f \frac{Dy}{Dt}$$

$$\int_{v(0)}^{v(t)} \frac{Dv}{Dt} dt = \int_{y(0)}^{y(t)} f \frac{Dy}{Dt} dt \Rightarrow v(t) = v_{g0} + f \delta y$$

$$\text{so } \frac{Dv}{Dt} = f \left(v_{g0} + \frac{\partial u_g}{\partial z} \delta z + \frac{\partial u_g}{\partial y} \delta y - v_{g0} - f \delta y \right) = - \left(f - \frac{\partial u_g}{\partial y} \right) \delta y$$

\Rightarrow favored by small $R_i \rightarrow$ larger $\frac{\partial u_g}{\partial z}$

$$\text{small } \frac{\partial g}{\partial z}$$

small S_g

along θ -sfc.

$$\left\{ \begin{array}{l} \frac{\partial \theta}{\partial t} = 0 \Rightarrow \text{adiabatic} \\ \frac{\partial w}{\partial t} = 0 \Rightarrow \text{hydrostatic} \end{array} \right.$$

What happens not along θ sfc.
vertically?
horiz?

\rightarrow suppose $\frac{\partial u_g}{\partial z} = 0 \Rightarrow \theta$ sfc. horiz.

$$\Rightarrow R_i = \infty \Rightarrow S_g + f < 0$$

\rightarrow suppose $\frac{\partial \theta}{\partial z} = 0 \Rightarrow R_i = 0 \Rightarrow$ all S_g 's unstable
i.e. θ sfc. vert

for ^{purely}
vertical displ. $\frac{\partial w}{\partial t} = \frac{\partial^2 (\delta_z)}{\partial t^2} = -\frac{g}{\theta} \frac{\partial \theta}{\partial z} \delta_z$

$$\sigma^2 = \frac{g}{\theta} \frac{\partial \theta}{\partial z} < 0$$

for ^{purely}
horiz displ. $\frac{\partial v}{\partial t} = \frac{\partial^2 (\delta_y)}{\partial t^2} = -f(S_g + f) \delta_y$

$$\sigma^2 = f(S_g + f) < 0$$

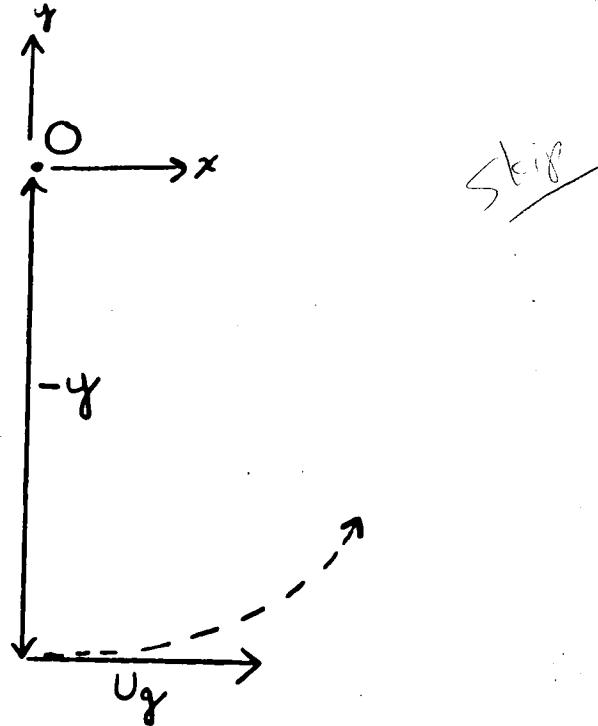
suppose

$$\frac{\partial \theta}{\partial z} > 0 \text{ stable for vert. displ.}$$

$$S_g + f > 0 \quad \text{" " horiz. displ.}$$

but $R_i < \frac{f}{S_g + f}$ symm unstable along θ sfc.

\Rightarrow stability depends upon direction of displacement



U_g = relative azimuthal velocity about point O, the origin

$f(-y)$ = rotation rate about local vertical due to Earth's rotation \times distance from axis of rotation = azimuthal velocity about point O due to Earth's rotation

$v_r = U_g - f(-y) = \text{absolute azimuthal velocity about point O}$

(1)

$$\frac{\partial U_g}{\partial z} = -g \frac{fT}{fT} \frac{\partial T}{\partial y} + \underbrace{\frac{U_g}{T} \frac{\partial T}{\partial z}}$$

small

$$\theta = T \left(\frac{p_0}{p} \right)^k \Rightarrow T = \theta \left(\frac{p}{p_0} \right)^k$$

$$= -g \frac{(p_0)^k}{f\theta} \left[\frac{\partial \theta}{\partial y} \left(\frac{p}{p_0} \right)^k + \theta \left(\frac{1}{p_0} \right)^k p^{k-1} \frac{\partial p}{\partial y} \right] + \left(\frac{U_g}{\theta} \left(\frac{p_0}{p} \right)^k \left[\frac{\partial \theta}{\partial z} \left(\frac{p}{p_0} \right)^k \right. \right.$$

$$\left. \left. \theta \left(\frac{p}{p_0} \right)^k \frac{k}{p} \frac{\partial p}{\partial y} \right] \right)$$

$$\left. \theta \left(\frac{1}{p_0} \right)^k k p^{k-1} \frac{\partial p}{\partial z} \right]$$

$$\approx -g \frac{f\theta}{f\theta} \left[\frac{\partial \theta}{\partial y} + k \theta \frac{\partial p}{p} \frac{\partial y}{\partial y} \right]$$

small

$$\sim \frac{10K}{10^5 m} \quad \sim \frac{0.3 \times 300K}{100 \text{ bars}} \frac{1 \text{ bar}}{10^5 m}$$

↑ ↑ or less

$$10^{-4} K m^{-1}$$

$$10^{-5} K m^{-1}$$

$$\boxed{\frac{\partial U_g}{\partial z} \approx -g \frac{f\theta}{f\theta} \frac{\partial \theta}{\partial y}}$$

~~(*)~~ Width of symmetrically unstable vertical circulation?

If along θ sta.

$$\left(\frac{dz}{dy}\right)_0 = \frac{f \frac{\partial u_g}{\partial z}}{\frac{g \frac{\partial \theta}{\partial z}}{\theta}} \sim \frac{H}{L}$$

$$\text{but } \left(\frac{dz}{dy}\right)_0 > \left(\frac{dz}{dy}\right)_{mg} = \frac{f - \frac{\partial u_g}{\partial y}}{\frac{\partial u_g}{\partial z}}$$

$$\text{so, } \frac{H}{L} > \frac{f - \frac{\partial u_g}{\partial y}}{\frac{\partial u_g}{\partial z}}$$

$$\frac{H \frac{\partial u_g}{\partial z}}{f - \frac{\partial u_g}{\partial y}} > L \Rightarrow L < \frac{\frac{\partial u_g}{\partial z} H}{f - \frac{\partial u_g}{\partial y}}, \quad H \sim \left(\frac{50 \text{ m s}^{-1}}{10^4 \text{ m}}\right) 10^4 \text{ m}$$

$\approx 500 \text{ km}$

mesoscale!

→ Useful to express nec. condx. for sym. instab. in terms of Erde's potential vorticity

$$\mathcal{Z} = (\vec{\nabla}_z \times \vec{V} + f \hat{k}) \cdot \vec{\nabla}_z s$$

$$s = c_p \ln \theta \quad (V_g = 0)$$

$$\mathcal{Z} = \frac{c_p}{\rho} \left[\frac{\partial u_g}{\partial z} \hat{j} + \left(f - \frac{\partial v_g}{\partial z} \right) \hat{k} \right] \cdot \left[\underbrace{\frac{1}{\theta} \frac{\partial \theta}{\partial y} \hat{j}}_{S_g + f} + \frac{1}{\theta} \frac{\partial \theta}{\partial z} \hat{k} \right]$$

Thermal wind balance $\rightarrow -\frac{f}{g} \frac{\partial u_g}{\partial z}$

$$\mathcal{Z} = \frac{c_p}{\rho} \left[-\frac{f}{g} \left(\frac{\partial u_g}{\partial z} \right)^2 + (S_g + f) \frac{1}{\theta} \frac{\partial \theta}{\partial z} \right]$$

$$= \left(\frac{c_p}{\rho g} \right) \frac{f g \frac{\partial \theta}{\partial z}}{R_i} \left[-\frac{1}{R_i} + \frac{S_g + f}{f} \right]$$

< 0 for sym. instab. $\Rightarrow \mathcal{Z} < 0$

if $\frac{f g}{R_i} > 0, S_g + f > 0$

express Z in isentropic coord.

$$V_g(y, z) \rightarrow dV_g = \frac{\partial V_g}{\partial y} dy + \frac{\partial V_g}{\partial z} dz$$

$$\left(\frac{\partial U_g}{\partial y} \right)_s = \left(\frac{\partial U_g}{\partial y} \right)_z + \left. \frac{\partial U_g}{\partial z} \right|_{y,y} (dy)_s$$

$$f \frac{\partial U_g}{\partial z}$$

$$\frac{g \cdot \partial \theta}{\theta \cdot \partial z}$$

so,

$$\underbrace{\left(f - \frac{\partial U_g}{\partial y} \right)_s}_{(f + S_g)_s} = \underbrace{\left(f - \frac{\partial U_g}{\partial y} \right)_z}_{(f + S_g)_z} - \underbrace{f \frac{\partial U_g}{\partial z}}_{\frac{g \cdot \partial \theta}{\theta \cdot \partial z}}$$

$$(f + S_g)_s - (f + S_g)_z$$

$$\frac{f}{R_i}$$

$$\text{Substitute } \frac{f}{(f + S_g)_s}$$

$$\text{Then } Z = \frac{C_p}{pg} f \frac{g \cdot \partial \theta}{\theta \cdot \partial z} \left[\frac{(S_g + f)_s + f/R_i}{f} - \frac{1}{R_i} \right]$$

$$(S_g + f)_s + f/R_i = \frac{f}{R_i}$$

$$f$$

$$= \frac{C_p}{pg} \frac{g \cdot \partial \theta}{\theta \cdot \partial z} (S_g + f)_s$$

$$>_0$$

\approx goes abs vort.

on $s \neq s_0$

< 0 for sym. instab.

Useful form!

Convert Z to p coordinates; used in Sawyer-Elliott eqn.

$$V_g(y, p)$$

$$dV_g = \frac{\partial V_g}{\partial y} dy + \frac{\partial V_g}{\partial p} dp$$

$$\left(\frac{\partial U_g}{\partial y} \right)_p = \left(\frac{\partial U_g}{\partial y} \right)_y + \left. \frac{\partial U_g}{\partial p} \right|_{y,y} (dp)_p$$

$$\left(f - \frac{\partial U_g}{\partial y} \right)_p = \left(f - \frac{\partial U_g}{\partial y} \right)_y - \left. \frac{\partial U_g}{\partial p} \right|_{y,y} (dp)_p$$

$$d\theta = \frac{\partial \theta}{\partial y} dy + \frac{\partial \theta}{\partial p} dp = 0 \quad \text{along } \theta \text{ stz.}$$

$$\left(\frac{\partial p}{\partial y} \right)_\theta = - \frac{\frac{\partial \theta}{\partial y}}{\frac{\partial \theta}{\partial p}}$$

thermal wind $\frac{\partial u_g}{\partial p} = \frac{R}{f p} \left(\frac{p}{p_0} \right)^K \frac{\partial \theta}{\partial y}$

$$\begin{aligned} \stackrel{s_0}{\rightarrow} \left(f - \frac{\partial u_g}{\partial y} \right)_g &= \left(f - \frac{\partial u_g}{\partial y} \right)_p + \frac{\partial u_g}{\partial p} \left(\frac{f_p}{R} \left(\frac{p_0}{p} \right)^K \frac{\partial \theta}{\partial p} \right) \\ \left(f - \frac{\partial u_g}{\partial y} \right)_g \frac{\partial \theta}{\partial p} &= \left(f - \frac{\partial u_g}{\partial y} \right)_p \frac{\partial \theta}{\partial p} + \frac{f_p}{R} \left(\frac{p_0}{p} \right)^K \left(\frac{\partial u_g}{\partial p} \right)^2 > 0 \end{aligned}$$

LO

for
sym
in E&B

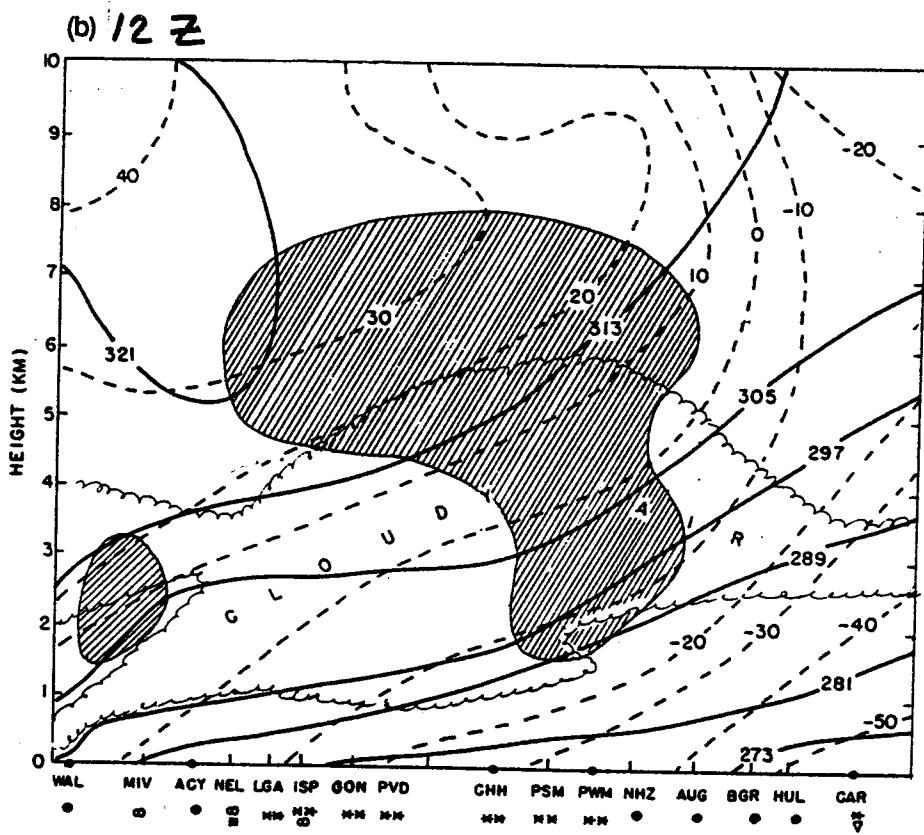
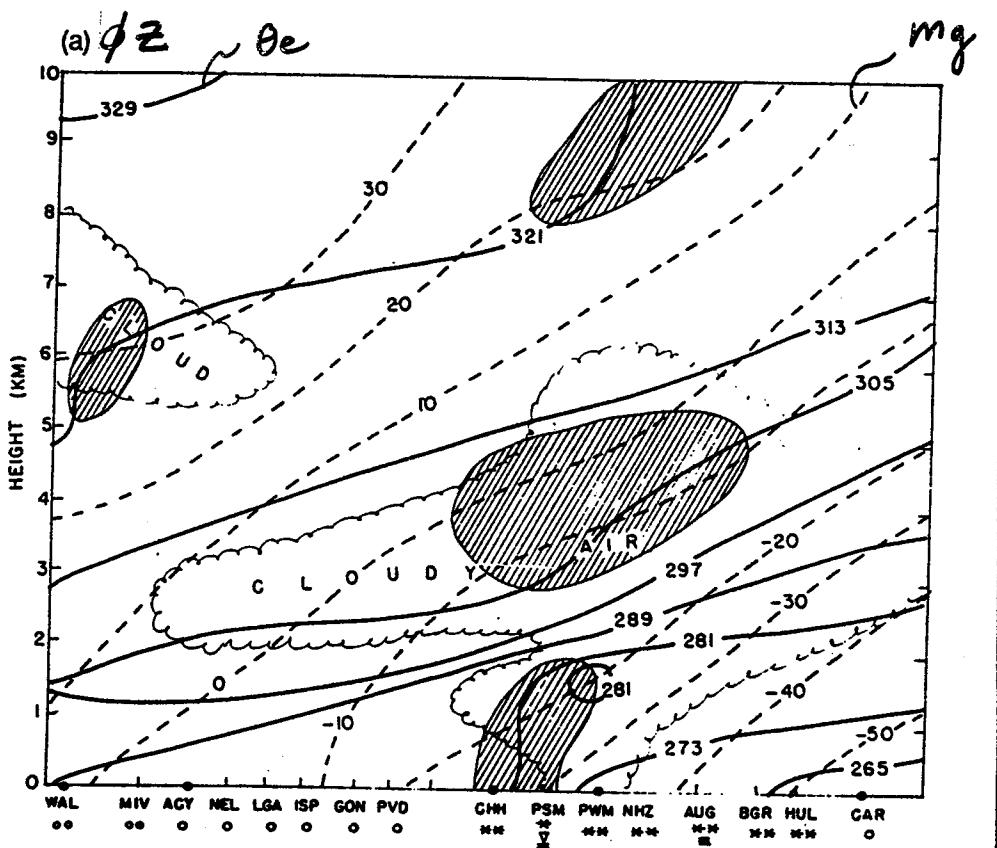
condition for
non ellipticity!

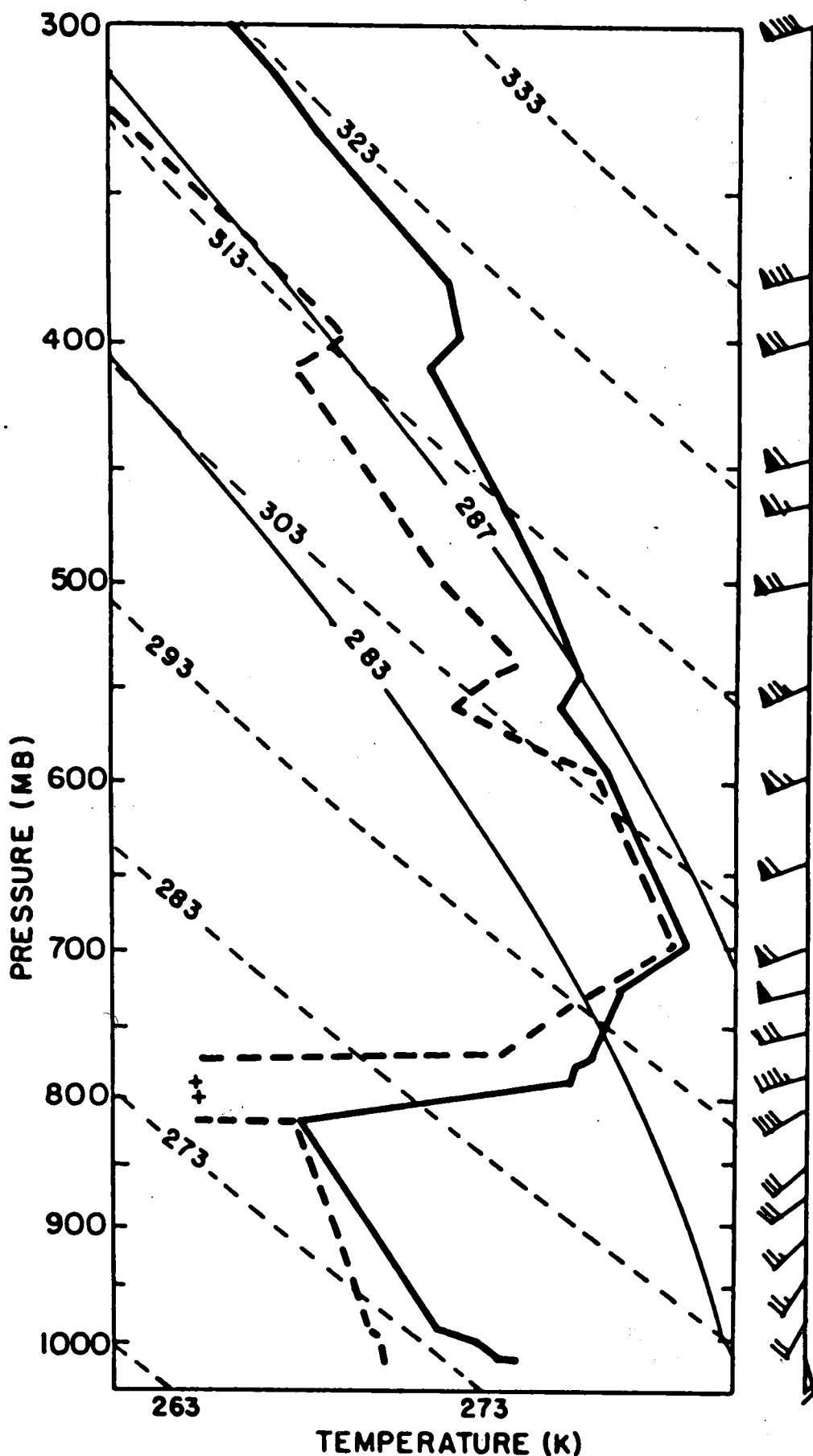
→ removal of instab by vert circ.

CSI

12/11/82

from Wolfsberg et al.
1986





from
Wolfsborg et
al. (1986)

FIG. 4. 00 GMT sounding at Chatham. Atmospheric temperature and dew-point temperature are shown in heavy solid and dashed lines. Reference pseudo-adiabats and moist adiabats are given in light solid and dashed lines. Winds are plotted along the right side.

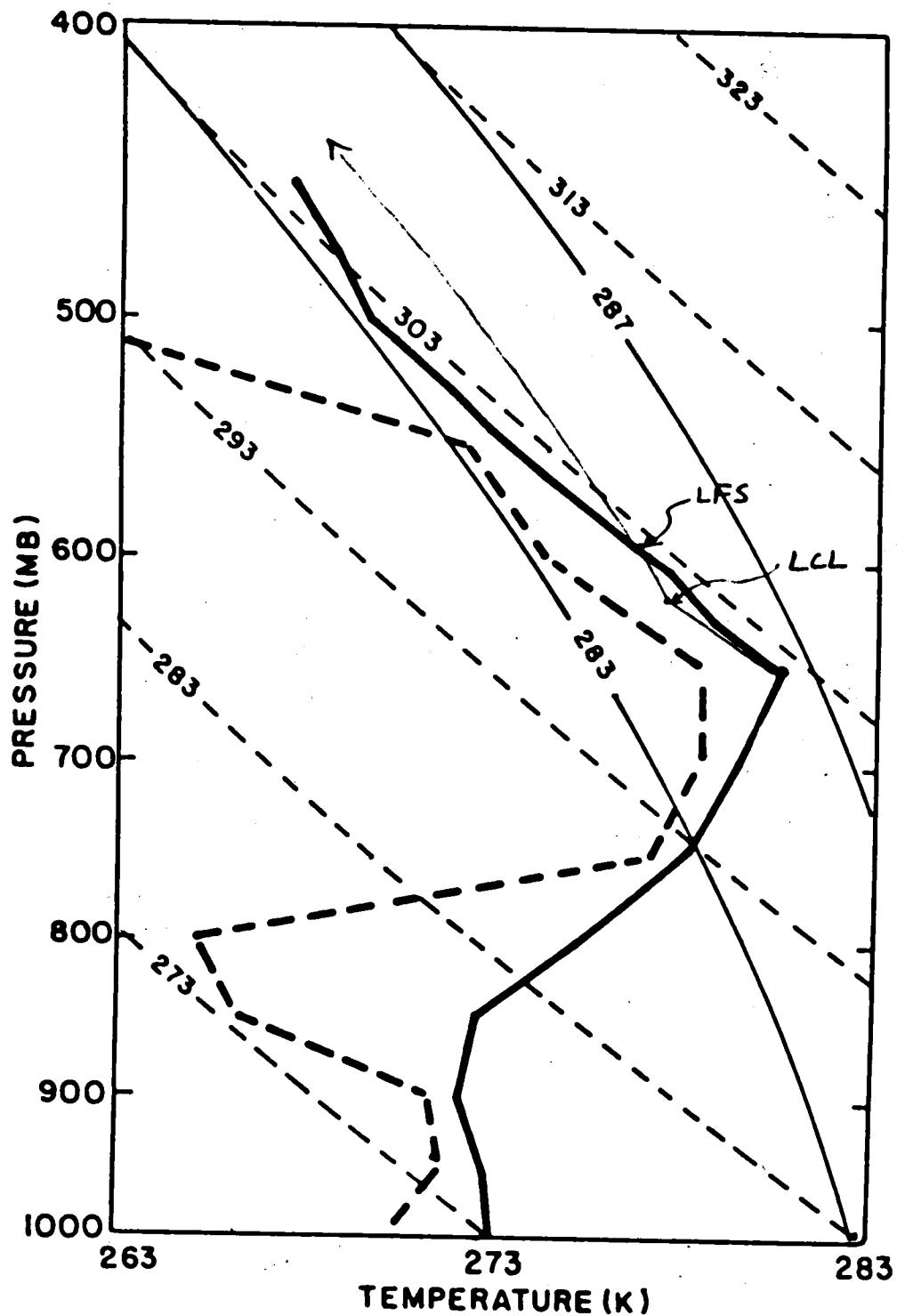


FIG. 11. Sounding along $M_g = -10 \text{ m s}^{-1}$ at 00 GMT. Potential temperatures and dew points were interpolated along the M_g -surface shown in Fig. 10a. Positive area can be interpreted as potential energy for displacements along the M_g surface.

after Wolfsberg et al.
(1986)