Review of the Quasi-Geostrophic System

In synoptic lecture, you have seen how scale analysis of the equations of motion leads to the following set of consistent, quasi-geostrophic equations:

<u>Vorticity Eq.</u> (neglect friction)

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla_p(\zeta_g + f) - f_o \nabla_p \vec{V}$$

or

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla_p \zeta_g - \beta V_g + f_o \frac{\partial \omega}{\partial p} \qquad \beta = \frac{df}{dy}$$

Vorticity changes are caused by advection of relative plus earth's vorticity and by divergence/convergence. (see Fig. 6.7 in Holton)

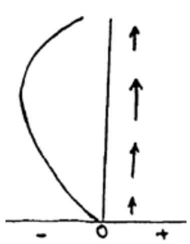
<u>Thermodynamic Eq.</u> (neglect diabatic term)

$$\frac{\partial T}{\partial t} = -\vec{V}_g \cdot \nabla_p T + \omega \frac{\sigma p}{R}$$

Temperature changes are caused only by *advection* and adiabatic *ascent/descent*.

<u>Mass Continuity Eq.</u> $\nabla_p \cdot \vec{V}_p T + \frac{\partial \omega}{\partial p} = 0$

Examples of physical interpretation:



$\frac{\partial \omega}{\partial p} < 0 \quad \therefore \nabla_p \vec{V} > 0$ horizontal divergence

$\frac{\partial \omega}{\partial p} > 0 \quad \therefore \nabla_p \vec{V} < 0$ horizontal convergence

$\omega(\mu bs^{-1})$ note: $\frac{\partial}{\partial p}\left(\frac{\partial \omega}{\partial p}\right) > 0 \Rightarrow \boxed{\omega < 0}$ rising motion

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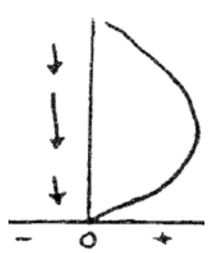
or,



$\frac{\partial \omega}{\partial p} > 0 \quad \nabla_p \vec{V} < 0$ horizontal convergence

$\frac{\partial \omega}{\partial p} < 0 \quad \nabla_p \vec{V} > 0$ horizontal divergence

ω and $\frac{\partial^2 \omega}{\partial p^2} < 0 \quad \therefore \boxed{\omega > 0}$ sinking motion

These 3 equations can be combined to help answer two of the primary questions of weather forecasting:

(1) Where are the highs and lows (ridges and troughs) going? That is, what are the *height changes* $\chi = \frac{\partial\varphi}{\partial t}$?

(2) Where is the precipitation going to be? i.e., What is the *distribution* of vertical motion ω?

The answer to (1) is found by examining the *height tendency equation:*

$$\left[\nabla_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\chi = f_o\left[-\vec{V}_g\cdot\nabla_p(\zeta_g+f)\right] - \frac{f_o^2}{\sigma}\frac{\partial}{\partial p}\left[\frac{R}{p}\left(-\vec{V}_g\cdot\nabla_p T\right)\right]$$

The answer to (2) is contained in the ω-equation.

$$\left[\nabla_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = -\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\cdot\nabla_p(\zeta_g+f)\right] - \frac{R}{\sigma p}\nabla_p^2\left[-\vec{V}_g\cdot\nabla_p T\right]$$

Both of the equations were derived in synoptic lecture. We shall first provide a mathematical interpretation of these equations followed by a physical interpretation. The mathematical interpretation is accomplished by the usual "inside-out" approach.

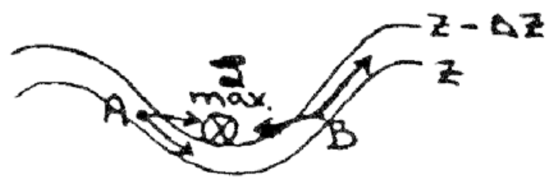
First consider the *vorticity advection term* in the height tendency equation:

$$\nabla_p^2\chi + \ldots\ldots = f_o\left[-\vec{V}_g\cdot\nabla_p(\zeta_g+f)\right]$$

We have seen earlier in this course that at point A, $\vec{V}_g\cdot\nabla_p\zeta_g > 0$, so $-\vec{V}_g\cdot\nabla_p(\zeta_g+f) < 0 \Rightarrow$ (anticyclonic vorticity advection)

Thus the right hand side is negative and $\nabla_p^2\chi < 0$. Since $\nabla^2(\;) < 0$ means that $(\;) \sim> 0$, we see that AVA leads to $\frac{\partial\varphi}{\partial t} > 0$ or *height rises*.

Similarly, at point B, $\vec{V}_g\cdot\nabla_p\zeta_g < 0$, $-\vec{V}_g\cdot\nabla p(\zeta_g+f) > 0$, (cyclonic vorticity advection) which leads to $\nabla^2\chi > 0$ and $\chi < 0$:*height falls*.

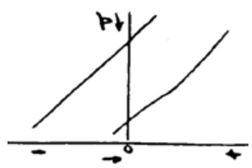


We now consider the *differential temperature advection* term:

$$\nabla_p^2 + ... = -\frac{f_o^2}{\sigma}\frac{\partial}{\partial p}\left[\frac{R}{p}\left(-\vec{V}_g \cdot \nabla_p T\right)\right]$$

Here we need to estimate temperature advection at two or more levels. For example, suppose we have the following vertical distributions of temperature advection:

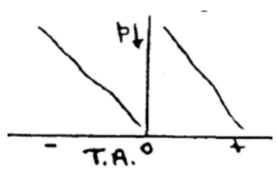
Thus $\frac{\partial}{\partial p}\left[\text{T.A.}\right] < 0$, $\nabla^2\chi > 0$, and $\chi < 0$: *height falls*.



$-\vec{V}g \cdot \nabla_p T$

Thus temperature advection *increasing* with height causes height falls. (Note that this can be caused by warm advection increasing with height or cold advection decreasing with height).

Similarly, temperature advection *decreasing* with height causes *height rises*. (true for a warm advection decreasing with height or cold advection increasing with height). (see Fig. 6.8 in Holton)

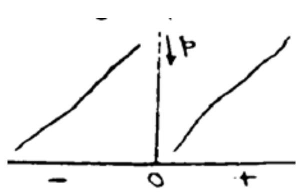


The mathematical interpretation of the *ω-equation* is as follows: Consider the *differential vorticity advection term*:

$$\nabla_p^2\omega + ... = -\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla_p\left(\zeta_g + f\right)\right]$$

Assume, e.g., that vorticity advection increases with height (true for both curves shown) Then

$\frac{\partial}{\partial p}(\text{V.A.}) < 0$ but $\frac{\partial}{\partial p}(\text{V.A.}) > 0$ and $\nabla^2\omega > 0$ which implies that $\omega < 0$: *rising motion*.



$-\vec{V} \cdot \nabla_p \zeta_g$

Therefore, vorticity advection *increasing* with height leads to upward motion.

Similarly, vorticity advection *decreasing* with height (both curves shown leads to *sinking motion*.). See Fig. 6.9 from Holton)

Consider the temperature advection term:

$$\nabla_p^2 \omega + ... = -\frac{R}{\sigma p} \nabla_p^2 \left[-\vec{V}_g \cdot \nabla_p T \right]$$

This says ω is large when gradients of temperature advection are large. If we have *warm advection*, then

$$-\vec{V}_g \cdot \nabla_p T > 0 , \quad \nabla^2 (T.A.) < 0 , \quad \nabla^2 \omega > 0$$

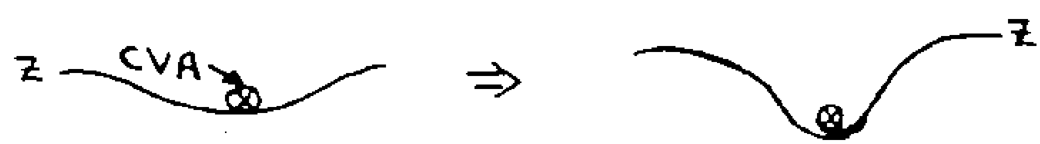
which implies $\omega < 0$ or *rising motion*. Similarly, *cold advection* leads to $\omega > 0$ or *sinking motion*. (see Fig. 6.10 from Holton)

Physical Interpretation of Quasi-Geostrophic Forcing Functions

Consider the height tendency (χ) equation and recall the *vorticity advection* term:

$$f_o \left[-\vec{V}_g \cdot \nabla_p (\zeta_g + f) \right]$$

If we have cyclonic vorticity advection (CVA) into a region, then the ζ_g increases there:



If vorticity is ≈ geostrophic (i.e., $-\zeta_g = \frac{g}{f_o} \nabla^2 z$), then we have height *decreases* as ζ_g increases. Or, in other words, larger vorticity values require stronger winds which require steeper pressure gradients which leads to height falls relative to the regions with weaker or no cyclonic vorticity advection.

Similarly, *AVA* requires *height rises*.

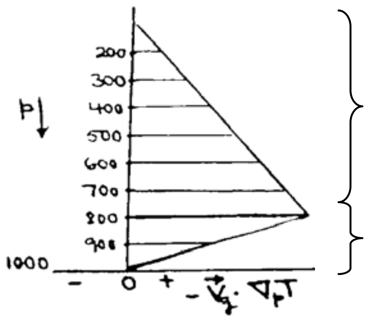
Consider the *differential thermal advection* term:

$$-\frac{f_o^2}{\sigma} \frac{\partial}{\partial p} \left[\frac{R}{p} \left(-\vec{V}_g \cdot \nabla_p T \right) \right]$$

Recall that we have shown that if:

temperature advection increases with height: *height falls*
temperature advection decreases with height: *height rises*
[don't have to use terms "warm" or "cold"]

For example, assume we have *warm advection* at *all levels*:



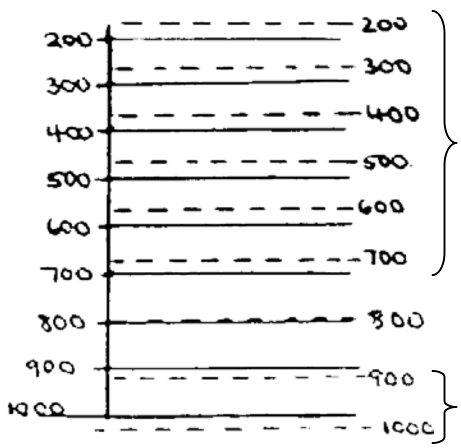
Mathematically, we see we have:

$$-\frac{\partial}{\partial p}(T.A.) < 0 \qquad \chi > 0 \Rightarrow \text{(rises)}$$

$$-\frac{\partial}{\partial p}(T.A.) > 0 \qquad \chi < 0 \Rightarrow \text{(falls)}$$

What is the physical reason for this? Recall that warm advection acts to warm each layer and will "thicken" each layer but will do so *most* where warm advection is the largest.

Now consider the old and new location of pressure surfaces after the warm advection shown above has acted for a while:



pressure surfaces pushed upwards – *height rises*

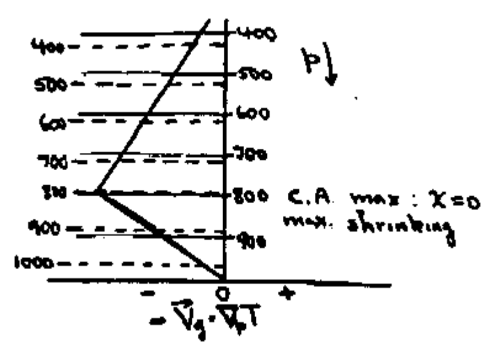
max. "thickening" occurs here

$$\therefore \frac{\partial}{\partial \mathrm{p}}(\mathrm{T.A.}) = 0 \qquad \rightarrow \chi = 0$$

pressure surfaces pushed downward – *height falls*

Thus warm advection causes height rises due to thicker layers above the level of maximum advection and height falls below this level due to hydrostatic warming of the column.

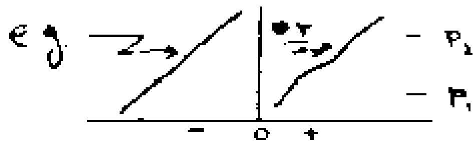
Similarly, cold advection produces both height rises beneath C.A. maximum due to increase in weight of the column and height falls above C.A. maximum due to accumulated effect of denser (thinner) layers.



Now consider the *differential vorticity advection* term of the *ω-equation*:

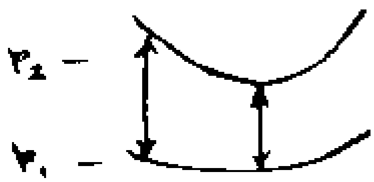
$$-\frac{fo}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla_p(\zeta_g + f)\right]$$

Why does *differential* vorticity advection cause vertical motion? Consider vorticity advection increasing with height:



(neglect temp. adv.)

If vorticity ≈ geostrophic, get (for curve on right)



- large ζ_g increase → large height fall

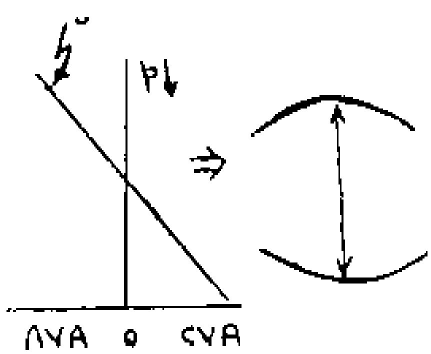
- small ζ_g increase → small height fall

Therefore, we have a *thickness decrease* where V.A. difference is greatest.

Thus the column must *cool* to keep atmosphere hydrostatic. Using temperature equation with $\frac{\partial T}{\partial t} < 0$, we see that upward motion and adiabatic cooling is required.

$$\frac{\partial \overline{T}}{\partial t} = \omega \frac{-\overline{\sigma p}}{R} \Rightarrow \omega < 0$$

Similarly, we get *sinking motion* if vorticity advection *decreases* with height.



Although the curve represents both positive and negative vorticity advection, we have sinking motion in both layers because adiabatic warming is required to explain the resulting thickness increase.

Finally, we consider the *laplacian of thermal advection* term in the ω-equation:

$$-\frac{R}{\sigma p}\nabla_p^2\left[-\vec{V}_g \cdot \nabla_p T\right]$$

If we have warm advection, we know $\omega < 0$ or rising motion. Why? If a layer experiences warm advection, its thickness will increase as shown below. This leads to a *relative* increase in vorticity at the bottom and a decrease at the top of the layer.

ζ_g decreases

ζ_g increases

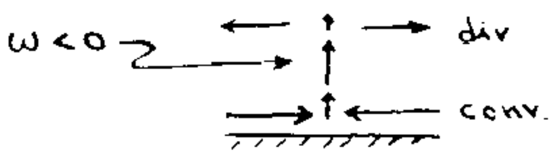
From the vorticity equation,

$$\frac{\partial \zeta_g}{\partial t} = -f_o D_{iv} \quad \text{(neglect vorticity adv.)}$$

Thus $\frac{\partial \zeta_g}{\partial t} > 0$ near the bottom $\rightarrow$ $D_{iv} < 0$: *convergence*

and $\frac{\partial \zeta_g}{\partial t} < 0$ aloft $\rightarrow$ $D_{iv} > 0$: *divergence*

From the mass continuity equation, we know convergence below and divergence above leads to rising motion:



Similarly, cold advection leads to subsidence.

TABLE 1: Idealized Q-G Description of a Mid-Latitude Baroclinic Wave**

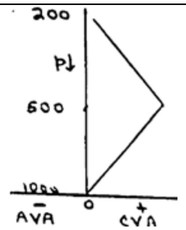
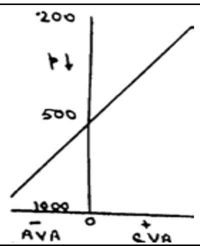
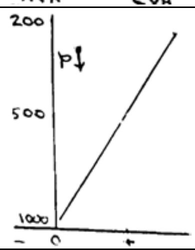
Parameter	**A: 500 mb trough**	**B: sfc low**	**C: 500 mb ridge**	**D: sfc high**
ζ_g-500: from vorticity eq.	Increasing due to convergence* (no VA)	Increasing due to CVA, partly cancelled by divergence	Decreasing due to divergence* (no VA)	Decreasing due to AVA partly cancelled by convergence
^{W}500: from QG ω eq.	Sinking due to CA (no VA)	Rising due to DIFF CVA (TA small)	Rising due to WAX (no VA)	Sinking due to DIFF AVA (TA small)
$^{\chi}$500: from QG χ eq.	Decreasing due to TA increasing with height	Decreasing due to CVA	Increasing due to TA decreasing with height	Increasing due to AVA
^{H}500-1000: from temp. eq.	Decreasing due to CA, partly cancelled by ad. Warming	Decreasing due to adiabatic cooling (TA small)	Increasing due to WA, partly cancelled by ad. Cooling	Increasing due to adiabatic warming (TA small)
ζg-sfc: from vorticity eq.	Decreasing due to divergence (VA small)	Increasing due to convergence (VA small)	Increasing due to convergence (VA small)	Decreasing due to divergence (VA small)

* assumes level of nondivergence is below 500 mb

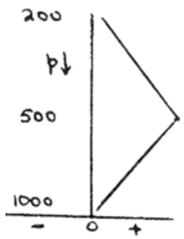
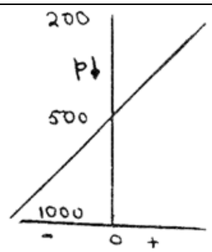
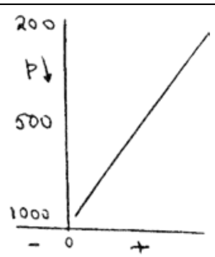
LEGEND:
VA : vorticity advection
TA : temperature advection
CVA : cyclonic vorticity advection
AVA : anticyclonic vorticity advection
CA : cold advection
WA : warm advection

** To accompany Fig. 6.11 in Holton, p. 141.

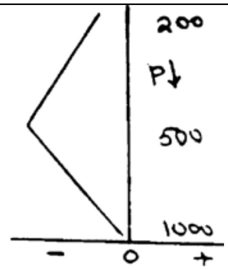
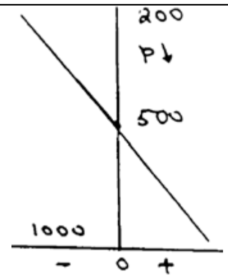
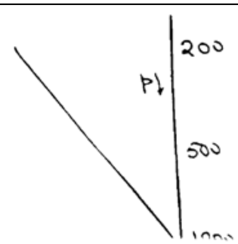
Relationships between Vorticity Advection and Response of χ, ω, D_{iv} and Thickness Fields (1)

Vertical Profile of Vorticity Adv.	Vorticity Change	QG Height Change (χ)	QG Vertical Motion (ω)	Divergence Response $\left(\frac{\partial \omega}{\partial p} = -D_{iv}\right)$	Thickness Response
	$\frac{\partial \zeta}{\partial t} > 0$ (but decreasing with height) $\frac{\partial \zeta}{\partial t} > 0$ (and increasing with height)	$\chi < 0$ (falls) $\chi < 0$ (falls)	$\omega > 0$ ↓ $\omega < 0$ ↑	↓ ←Div→ ↑ ↑ →Conv← ↑	Thickness increases (warming) Thickness decreases (cooling)
	$\frac{\partial \zeta}{\partial t} > 0$ (and increasing with height) $\frac{\partial \zeta}{\partial t} < 0$ (and increasing with height)	$\chi < 0$ (falls) $\chi > 0$ (rises)	$\omega < 0$ ↑ $\omega < 0$ ↑	↑ ←Div→ ↑ ↑ →Conv← ↑	Thickness decreases (cooling) Thickness decreases (cooling)
	$\frac{\partial \zeta}{\partial t} > 0$ (and increasing with height) $\frac{\partial \zeta}{\partial t} > 0$ (and increasing with height)	$\chi < 0$ (falls) $\chi < 0$ (falls)	$\omega < 0$ ↑ $\omega < 0$ ↑	↑ ←Div→ ↑ ↑ →Conv← ↑	Thickness decreases (cooling) Thickness decreases (cooling)

Relationship between Temp. Adv. And Response of Height, ω, D_{iv}, and ζ Fields

Vertical Profile of Temp. Advection	**Temperature Change**	**QG Height Change**	**QG Vertical Motion**	**Divergence Response** $\left(\frac{\partial\omega}{\partial p} = -D_{iv}\right)$	**Vorticity Response** $\left(\frac{\partial\zeta}{\partial t} \propto -f_o D_{iv}\right)$
	T.A. decreasing with height	rise	$\omega < 0$ ↑	↑ ←Div→ ↑	$\frac{\partial\zeta}{\partial t} < 0$
	max. warming	------------	(max) ---$\omega < 0$--- ↑	----↑----	------------
	T.A. increasing with height	fall	$\omega < 0$ ↑	→Conv← ↑	$\frac{\partial\zeta}{\partial t} > 0$
	warning (increases with height)	fall	$\omega < 0$ ↑	↑ →conv. ←	
	------------	------------	---- $\omega = 0$ ----	----------------	----$\frac{\partial\zeta}{\partial t} > 0$----
	cooling (decreases with height)	fall	$\omega > 0$ ↑	↓	
	max. warming	fall	$\omega << 0$ ↑	↑ →conv. ← ↑	$\frac{\partial\zeta}{\partial t} > 0$
	T.A. increases with height	------------	↑	↑	
	Small warming	fall	$\omega < 0$ ↑	→conv.← ↑	$\frac{\partial\zeta}{\partial t} > 0$

Relationships between Temp. Adv. And Response of Height, ω, D_{iv}, and ζ Fields (cont.)

Vertical Profile of Temp. Advection	Temperature Change	QG Height Change	QG Vertical Motion	Divergence Response $\left(\frac{\partial\omega}{\partial p} = -D_{iv}\right)$	Vorticity Response $\left(\frac{\partial\zeta}{\partial t} \propto -f_o D_{iv}\right)$
	T.A. increases with height large cooling T.A. decreases with height	falls ------------ rises	$\omega > 0$ ↓ $\omega >> 0$ ↓ $\omega > 0$ ↓	↑ →Conv← ↓ ←Div→ ↓	$\frac{\partial\zeta}{\partial t} > 0$ ------------ $\frac{\partial\zeta}{\partial t} < 0$
	cooling Temp. advection decreases with height warming	rises rises	$\omega > 0$ ↓ $\omega = 0$ ↓ $\omega < 0$ ↑	↓ ←Div→ ↓ ←----------→ ↑ ←→ ↑	$\frac{\partial\zeta}{\partial t} > 0$
	Large cooling Temp. adv. decreases with height	rises rises	$\omega >> 0$ ↓ $\omega > 0$ ↓ $\omega \sim 0$ ↓	↓ ← → ↓ Div ←----→ ↓	$\frac{\partial\zeta}{\partial t} < 0$ $\frac{\partial\zeta}{\partial t} << 0$