Problem Set for NWP and DA sections for METR 5004

Reading Assignment:

Chapter 1, Chapter 3, sections 1&2, Arakawa and Messinger, Numerical Methods used in Atmospheric Models GARP #17 1976 (should have this document as an attachment). You are responsible for this material.

Also good reference: Chapter 2, section 2.5.2-2.5.3, Numerical Methods for Wave Equations in Geophysical Fluid Dynamics, D. Durran.

P1. A well-known advection scheme is called the Lax-Wendroff or the Crowley scheme. Its finite difference form in 1-dimension is given as:

$$f_j^{n+1} = f_j^n - \frac{c\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(f_{j+1}^n - 2f_j^n + f_{j-1}^n\right)$$

where “c” is the a constant velocity advection.

a) Plug in a Talyor series expansion for (1) and determine the order of accuracy in space and time for this scheme.
b) Find the amplification matrix and show what conditions are necessary for stability.
c) If the analytical phase speed is given by

$$\theta_a = -kc\Delta t$$

derive an expression for the ratio of the numerical phase speed:

$$\frac{\theta_{LW}}{\theta_a}$$

d) for a 4 Δx wave, c = 0.25 and Δt = Δx = 1.0, calculate the phase speed ratio above.

P2. Using a least squares approach, determine the best estimate of the temperature and its uncertainty at a point where the temperatures and instrument uncertainties from two different instruments measuring at the same location is

T1 = 306.2 K	Stddev = 1.0 K
T2 = 308.3 K	Stddev = 1.75 K

Solutions to Problem Set for NWP and DA sections for METR 5004

P1

a) should be able to show that the two leading terms are

$$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}=O\left(\Delta t^2,\Delta x^2\right)$$

solutions to b-d

a) Crowley scheme $f_j^{n+1} = f_j^n - \frac{c_r}{2}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{c_r^2}{2}\left(f_{j+1}^n + f_{j-1}^n - 2f_j^n\right)$ $\quad C_r = \frac{u\Delta t}{\Delta x}$

let $f_j^n = A^n e^{ikj\Delta x}$, after substitution;

$$A = 1 - ic_r \sin k\Delta x + c_r^2\left(\cos k\Delta x - 1\right)$$

$$|A| = \left[R_e^2 + I_m^2\right]^{1/2} = \left[\left(1+c_r^2\left(\cos k\Delta x - 1\right)\right)^2 + c_r^2 \sin^2 k\Delta x\right]^{1/2}$$

easiest thing is to check various $k\Delta x$'s

$L = 2\Delta x$ $\quad k\Delta x = \pi \quad |A| = \left[\left(1-2c_r^2\right)^2\right]^{1/2} = 1-2c_r^2 \quad |A| \le 1$ for all

$L = 4\Delta x$ $\quad k\Delta x = \pi/2 \quad |A| = \left[\left(1-c_r^2\right)^2 + c_r^2\right] = 1 - c_r^2 + c_r^4 \quad |A| \le 1$ for $c_r^2 \le 1$

$L = \infty$ $\quad k\Delta x = 0 \quad |A| = 1$ The stability condition that is most restrictiv is $k\Delta x = \pi/2$ ($4\Delta x$ waves). Therefore $|c_r| \le 1$ is the CFL cond

b) $$\frac{\theta_d}{\theta_a} = \frac{\tan^{-1}\left[\frac{A_I}{A_R}\right]}{-kc\Delta t} = \frac{\tan^{-1}\left[\frac{-c_r \sin k\Delta x}{1+c_r^2\left(\cos k\Delta x - 1\right)}\right]}{-kc\Delta t}$$

(c) $$kc\Delta t = k\Delta x\, c_r = \frac{\pi}{2}\cdot\frac{1}{4} = \pi/8$$

$$\frac{\theta_d}{\theta_a} = \frac{\tan^{-1}\left[\frac{-\frac{1}{4}\sin\left(\pi/2\right)}{1+\left(\frac{1}{16}\right)(-1)}\right]}{\pi/8} = .6636$$

So $4\Delta x$ moves at 66% of analytical speed.

P2

Answer is T = 306.7102 K, Stddev = 1.1489 K